Graded modal logic with a single modality

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Graded modal logic is an extension of classical modal logic with graded modalities $\Diamond_n (n \in \mathbb{N}^+)$ that allows to count the number of successors of a given state in a Kripke model. Intuitively, the formula $\Diamond_n A$ is satisfied at a point w of a Kripke frame if and only if w has at least n successors satisfying A.

Graded modal logic was originally introduced in Goble [10]. Kaplan [12] studied graded modal logic as an extension of **S5**. The completeness of graded modal logic and its extensions was investigated in [9, 7, 2]. Van der Hoek [15] and Cerrato [3] used filtrations to obtain the finite model property and decidability of graded modal logic. Van der Hoek [15] also studied the expressibility, definability and correspondence theory. Bisimulations for graded modal logic were introduced in [8], and used to provide an alternative proof of the finite model property, and show that a first-order formula is invariant under graded bisimulation iff it is equivalent to a graded modal formula. Aceto, Ingolfsdottir and Sack [1] showed that resource bisimulation and graded bisimulation coincide over image-finite Kripke frames. Finally, various notions of epistemic and dynamic graded modal logics have been investigated in [16] and [13].

Even though the modality $\Diamond_1$ corresponds to the standard classical modal logic connective, and therefore retains all its properties, the modalities $\Diamond_n$ for $n \geq 2$ do not. In particular, the modalities $\Diamond_n$ are monotone, i.e. they satisfy the rule $\vdash \varphi \to \psi / \vdash \Diamond_n \varphi \to \Diamond_n \psi$, and satisfy $\Diamond_n \bot \leftrightarrow \bot$, but are not **additive**, that is, the implication $\Diamond_n(p \vee q) \to (\Diamond_n p \vee \Diamond_n q)$ fails for $n \geq 2$. Modal logics with monotone modalities have been extensively studied [4, 11, 14]. However, not much work has been done regarding the connections between monotonic modal logics and graded modal logic. In [6], building on the proof-theoretic and algebraic analysis of non-normal modal logics of [5], a line of research studying these connection was initiated, where an elementary but not modally definable class of neighbourhood frames was shown to exactly correspond to graded Kripke frames, and the notion of graded bisimulation was recasted through the lens of neighbourhood bisimulations.

This presentation reports on work that adds to the study of connections between monotonic modal logic and graded modal logic, albeit towards a different direction. Specifically, the standard axiomatization of graded modal logic relies on the **interaction** of the different graded modalities, and captures the properties of addition of natural numbers. However, when viewed as monotone modalities, each graded modality can also be studied in isolation. Accordingly, for every $n \in \mathbb{N}^+$, we introduce the logic $\mathcal{L}_n$, whose language contains a single modal operation, $\Diamond$, and whose theory is defined as the set of validities on Kripke frames, where $\Diamond$ is interpreted as the graded modality $\Diamond_n$ described in the first paragraph. We show that they have the finite model property, are decidable, and are finitely axiomatizable. We also show that for $n \neq m$ the logics $\mathcal{L}_n$ and $\mathcal{L}_m$ are distinct and in particular:

Theorem 1. *Let $n < m$ such that $m - 1 = (n - 1) \cdot k + r$ where $r < n - 1$. Then, if $r < k$ it follows that $\mathcal{L}_m \subsetneq \mathcal{L}_n$. If $k \leq r$, then there exists $\zeta_n, \theta_n \in \Phi$, such that $\zeta_n \in \mathcal{L}_n$ while $\zeta_n \notin \mathcal{L}_m$ and $\theta_n \in \mathcal{L}_m$ while $\theta_n \notin \mathcal{L}_n$.*

We will also discuss complete axiomatizations for these logics. In particular, assuming that α_1, α_2, α_3 are mutually contradictory and likewise for β_1 and β_2, and denoting

$$\Diamond_1^{\psi} \varphi := \Diamond(\varphi \vee \psi) \wedge \neg \Diamond \psi, \tag{1}$$

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the following theorems hold:

Theorem 2. *The system consisting of all propositional tautologies, the monotonicity rule*

$$(M) \quad \vdash (p \to q) / \vdash (\Diamond p \to \Diamond q),$$

and the following axioms

$$\begin{array}{rl} (\bot) & \Diamond\bot \to \bot, \\ (G2) & \left[\Diamond_1^{q_1}(\alpha_1) \wedge \Diamond_1^{q_2}(\alpha_2)\right] \to \Diamond(\alpha_1 \vee \alpha_2), \end{array}$$

and closed under modus ponens and uniform substitution is a sound and complete axiomatization of $\mathcal{L}_2$.

Theorem 3. *The system consisting of all propositional tautologies, the monotonicity rule*

$$(M) \quad \vdash (p \to q) / \vdash (\Diamond p \to \Diamond q),$$

and the following axioms

$$\begin{array}{rl} (\bot) & \Diamond\bot \to \bot, \\ (G3_1) & \left[\Diamond_1^{q_1}(\alpha_1) \wedge \Diamond_1^{q_2}(\alpha_2) \wedge \Diamond_1^{q_3}(\alpha_3)\right] \to \Diamond(\alpha_1 \vee \alpha_2 \vee \alpha_3), \\ (G3_2) & \left[\Diamond_1^{q_1}(\alpha_2) \wedge \Diamond_1^{q_2}(\beta_2) \wedge \Diamond(\alpha_1 \vee \beta_1) \wedge \neg\Diamond(\alpha_1 \vee \alpha_2)\right] \to \Diamond(\beta_1 \vee \beta_2), \end{array}$$

and closed under modus ponens and uniform substitution is a sound and complete axiomatization of $\mathcal{L}_3$.

Finally we will discuss possible techniques for obtaining axiomatizations for $\mathcal{L}_n$ for $n \geq 4$.

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