Equidivisibility and profinite coproduct

Alfredo Costa[1,*] and Jorge Almeida[2]

[1] University of Coimbra, CMUC, Department of Mathematics, Apartado 3008, EC Santa Cruz, 3001-501 Coimbra, Portugal.
amgc@mat.uc.pt

[2] CMUP, Departamento de Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal.
jalmeida@fc.up.pt

A semigroup is *equidivisible* if any two factorizations of an arbitrary element of the semigroup have a common refinement. The property of a semigroup being equidivisible was introduced and studied in [6] as a natural common generalization of free semigroups and groups.

More recently, this property appeared frequently as a useful tool in the study of relatively free profinite semigroups. A profinite semigroup is relatively free if it is a free object in the category of pro-V semigroups, for some pseudovariety V of finite semigroups. (A pseudovariety of semigroups is a class of finite semigroups closed under taking homomorphic images, subsemigroups, and finite products.)

In [1, 5] it was shown that if the pseudovariety V is such that the product of any two V-recognizable languages is still V-recognizable, then the finitely generated free pro-V semigroups are equidivisible. This includes the cases where V is the pseudovariety of all finite semigroups, and where V is the pseudovariety of all finite aperiodic semigroups. Other recent papers where the equidivisibility of relatively free profinite semigroups is applied or deserves attention include [4, 3, 8].

A complete characterization of the pseudovarieties for which the corresponding finitely generated relatively free profinite semigroups are equidivisible (dubbed *equidivisible pseudovarieties*) appears in [2]. This characterization is done via a functor on the category of finitely generated semigroups, called the *two-sided Karnofsky–Rhodes expansion*. Given an onto homomorphism $\varphi\colon A^+ \to S$, its two sided Karnofsky–Rhodes expansion is the onto homomorphism $\varphi^{\mathsf{KR}}\colon A^+ \to S_\varphi^{\mathsf{KR}}$ that identifies words with the same image under φ and such that the naturally associated paths of the two-sided Cayley graph of φ have the same transition edges. The semigroup S_φ^{KR} is said to be a two-sided Karnofsky–Rhodes expansion of S.

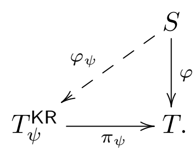
Theorem 1 ([2])**.** *A finitely generated profinite semigroup is equidivisible if and only if it is contained in the pseudovariety of completely simple semigroups or it is closed under two-sided Karnofsky–Rhodes expansion.*

As observed in [6], the class of equidivisible semigroups is closed under taking free products, that is, coproducts in the category of semigroups. Here we present an analog for profinite semigroups. For that purpose, we introduce V-coproducts of pro-V semigroups with respect to a pseudovariety of semigroups V, extending what was done in [7] for the pseudovariety of finite groups. We give simple conditions on V guaranteeing that the free product of pro-V semigroups embeds naturally in their V-coproduct.

The following definition is the key to obtain our main new results.

*Speaker.

Definition 2 (KR-cover of a finite semigroup)**.** Let S be a profinite semigroup, and let T be a finite semigroup. We say that S is a *KR-cover of* T when T is a continuous homomorphic image of S and for every continuous onto homomorphism $\varphi : S \to T$ there is a generating mapping $\psi : A \to T$, for some finite alphabet A depending on φ, and a continuous homomorphism $\varphi_\psi : S \to T_\psi^{\mathsf{KR}}$ such that the following diagram commutes:



A profinite semigroup S is a *KR-cover* if it is a KR-cover of each of its finite continuous homomorphic images.

As examples of KR-covers, we have the profinite groups and the free profinite semigroups relatively to a pseudovariety closed under taking two-sided Karnofsky–Rhodes expansion. We show that every KR-cover is equidivisible. One of our mains results is the following:

Theorem 3. *For every pseudovariety of semigroups* V *closed under two-sided Karnofsky–Rhodes expansion, the class of all pro-*V *KR-covers is closed under* V*-coproducts.*

This theorem allows us to build new examples of equidivisible profinite semigroups from old ones. However, there are examples of finite equidivisible semigroups that are not KR-covers.

Let A be a finite alphabet, and V be a pseudovariety closed under two-sided Karnofsky–Rhodes expansion. Using equidivisibility, one sees that the A-generated relatively free profinite semigroup over V, denoted $\overline{\Omega}_A\mathsf{V}$, has the following cancellation property: if $au = bv$ or $ua = vb$, with $a, b \in A$ and $u, v \in \overline{\Omega}_A\mathsf{V}$, then $a = b$ and $u = v$. Abstracting this property, we get the class of the so called *letter super-cancellative* profinite semigroups. It turns out that within this class the KR-covers completely characterize the equidivisible profinite semigroups. That is, we have:

Theorem 4. *Let* S *be a finitely generated profinite semigroup that is letter super-cancellative. Then* S *is equidivisible if and only if it is a KR-cover.*

We build examples of letter super-cancellative equidivisible profinite semigroups that are not relatively free profinite semigroups.

With the previous theorem on hand, we are able to deduce the following result.

Theorem 5. *For every pseudovariety of semigroups* V *closed under two-sided Karnofsky–Rhodes expansion, the class of letter super-cancellative equidivisible finitely generated pro-*V *semigroups is closed under finite* V*-coproducts.*

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