Initiating descent theory for closure spaces

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By a closure space we mean a set equipped with a set of its subsets closed under arbitrary intersections. We describe various classes of maps of closure spaces that naturally occur in their descent theory, including a complete characterization of effective descent morphisms of finite closure spaces, which is our main result. It turns out that the analogy with such descriptions in the case of preorders [2] is stronger than in the intermediate case of topological spaces (see [1] and [7]). Nevertheless various general (category-theoretic and) topological tools used in [4], [10], [5], [8], and [9] turn out to be helpful also in the context of closure spaces. We also briefly consider connections with what we called strict monadic topology in [3].

Note that several different notions of a closure space have been introduced by various authors, some of them a long time ago; and it is interesting that the recent paper [6] almost enters descent theory: among many other things it describes pullback stable regular epimorphisms of closure spaces, slightly different ones, but that result can be copied in our context.

In finitely complete categories with coequalizers of equivalence relations we isolate the conditions that make a descent morphism to fail to be effective and translate them in terms of closure spaces. Then, the main result is based on the following observation specific to closure spaces, which has no counterpart neither for general topological spaces nor for preorders:

Let E and E' be closure spaces with the same underlying set such that the identity map 1_E is a continuous map from E' to E, or, equivalently,

$$\overline{S}^{E'} \subseteq \overline{S}^{E} \text{ for all } S \subseteq E;$$

let $p : E \to B$ also be a continuous map of closure spaces. Then the following conditions are equivalent:

(a) the triple $(E', 1_E, \pi_1)$, where $\pi_1 : E \times_B E' \to E'$ is defined as the first projection, that is, by $\pi_1(e, e') = e$, is a descent data for p;

(b) $\overline{S}^{E} \cap p^{-1}p(\overline{p^{-1}p(S)}^{E'})) \subseteq \overline{S}^{E'}$ for all $S \subseteq E$.

Suppose the equivalent conditions (a) and (b) above are satisfied and let us write p' for p considered as a morphism from E' to B. If both p and p' are regular epimorphisms, then, for every set X closed as a subset of E' but not as a subset of E, there exists a set Y with the same property and $X \subset Y$ (strict inclusion).

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