Kan-injectivity and KZ-doctrines

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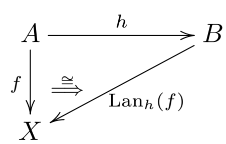
In a category of algebras for a given finitary signature, the reflection of an algebra A into a subcategory presented by a set of finitary equations (or implications) may be constructed by means of a transfinite chain of morphisms which converges at step ω. This construction may be performed in any locally presentable category, leading to a solution of the Orthogonal Subcategory Problem for a set of morphisms, as it was proved by Gabriel and Ulmer [9]. Thus, given a set $\mathcal{H}$ of morphisms, we conclude that the full subcategory of orthogonal objects constitutes the category of Eilenberg-Moore algebras of an idempotent monad. This result can be extended to a broader context, including locally bounded categories, as shown in [11].

In [6], Banaschewski and Herrlich observed that, for an algebra, the satisfaction of an implication is equivalent to the injectivity of the algebra with respect to a certain morphism. The combination of this idea with the properties of the above transfinite chain gave rise to the study of deduction systems where the "formulas" are morphisms, see [1], [2], [3].

In a 2-category $\mathcal{K}$, an object X is said to be *left Kan-injective* with respect to a morphism $h : A \to B$ if, for every morphism $f : A \to X$, there is a left Kan extension of f along h given by an invertible 2-cell:



A morphism $u : X \to Y$ is said to be *left Kan-injective* with respect to $h : A \to B$ if it preserves left Kan extensions of morphisms $f : A \to X$ along h, that is, $\mathrm{Lan}_h(uf) = u\mathrm{Lan}_h(f)$. We denote by $\mathrm{LInj}(\mathcal{H})$ the locally full subcategory of $\mathcal{K}$ of all objects and all morphisms left Kan injective with respect to every morphism of $\mathcal{H}$.

In 2-categories, certain conditions on objects, and on morphisms, may be given by means of left Kan injectivity. For instance, in the 2-category **Pos**, made of posets, monotone maps and pointwise order between maps, the posets with binary suprema and the morphisms which preserve them are precisely those which are left Kan-injective with respect to the embedding of the discrete poset D with two elements into the poset obtained from D by joining a top element. Taken in the 2-category **Cat** of categories, this embedding presents, via left-Kan-injectivity, the categories with binary coproducts and morphisms preserving them. Many other examples may be find in [5], [7], [8], [10] and [13].

Starting from a set $\mathcal{H}$ of morphisms in a 2-category $\mathcal{K}$, we construct, for each object X, a transfinite chain leading to the components of the unit of a lax-idempotent pseudo-monad (or KZ-doctrine, see [11] and [12]). The algebras and the homomorphisms of this KZ-doctrine are,

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essentially, the objects and the morphisms of $\mathcal{K}$ which are left Kan-injective with respect to $\mathcal{H}$. This encompasses, as a particular case, the transfinite chains mentioned above, and generalizes the Kan-injective reflection chain presented in [5] for order-enriched categories, which, in [4], was a key tool for obtaining a Kan-injectivity logic.

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