On varieties of residuated po-magmas and the structure of finite ipo-semilattices

PETER JIPSEN AND MELISSA SUGIMOTO

Chapman University, California, USA jipsen@chapman.edu
Leiden University, Netherlands, msr2236@gmail.com

In [6] *po-algebras* are defined as partially ordered sets with operations that are either order-preserving or order-reversing in each argument, and a variety of po-algebras is a class of similar po-algebras defined by equations or inequations.

A *residuated partially ordered magma* or *rpo-magma* $\mathbf{A} = (A, \leq, \cdot, \backslash, /)$ is a partially-ordered set $(A, \leq)$ with a binary operation $\cdot$ and two *residuals* that satisfy for all $x, y, z \in A$

$$\text{(res)} \qquad xy \leq z \iff x \leq z/y \iff y \leq x\backslash z$$

The operation $x \cdot y$ is usually written xy. Residuation ensures that $x\backslash y$ and y/x are order-reversing in the denominator (x position) and order-preserving in the numerator, while xy is order-preserving in both arguments. Since (res) is equivalent to the inequations $x \leq xy/y$, $(z/y)y \leq z$, $y \leq x\backslash xy$, $x(x\backslash z) \leq z$ it follows that rpo-magmas are a variety of po-algebras. Although rpo-magmas are very general, (res) imposes restrictions on the posets that can occur.

Theorem 1. *In an rpo-magma every connected component of $\leq$ is up-directed and down-directed, hence for finite rpo-magmas every connected component is bounded.*

The equivalence relation on a poset that has each connected component as an equivalence class is a congruence on a rpo-magma, and the quotient algebra is a quasigroup with the discrete order (i.e. $\leq$ is the equality relation). Conversely, from any quasigroup Q and a pair-wise disjoint family of *bounded* posets A_q for $q \in Q$, one can construct an rpo-magma with poset $\bigcup_{q \in Q} A_q$.

A *rpo-semigroup* or *Lambek algebra* is a rpo-magma where $\cdot$ is associative. A *unital* rpo-magma has a constant 1 such that $x1 = x = 1x$, and a *rpo-monoid* is a unital rpo-semigroup. A *residuated lattice-ordered magma* $(A, \wedge, \vee, \cdot, \backslash, /)$ (or *$r\ell$-magma* for short) is a rpo-magma for which the partial order is a lattice order. A $r\ell$-monoid is more commonly called a *residuated lattice.*

An *involutive partially-ordered magma* or *ipo-magma* is of the form $(A, \leq, \cdot, {\sim}, -)$ such that $(A, \leq)$ is a poset, $\cdot$ is a binary operation, the *left and right linear negations* ${\sim}, -$ are an *involutive pair*, i.e., ${\sim}{-}x = x = -{\sim}x$, $x \leq {\sim}y \iff y \leq -x$, and for all $x, y, z \in A$

$$\text{(ires)} \qquad xy \leq z \iff x \leq -(y \cdot {\sim}z) \iff y \leq {\sim}(-z \cdot x).$$

It follows that ${\sim}, -$ are both order-reversing. The axiom (ires) shows that every ipo-magma is term-equivalent to a rpo-magma, but it is often convenient to use the equivalent formulation

$$\text{(rotate)} \qquad xy \leq z \iff y \cdot {\sim}z \leq {\sim}x \iff -z \cdot x \leq -y.$$

The variety of ipo-monoids includes all partially ordered groups [1], where ${\sim}x = -x = x^{-1}$.

Lemma 2. *Let $\mathbf{A} = (A, \leq, \cdot, {\sim}, -)$ be a poset with a binary and two unary operations. (1) If $\cdot$ is idempotent (i.e. $xx = x$) and $\mathbf{A}$ satisfies* (rotate) *then $\mathbf{A}$ is an ipo-magma. (2) If an ipo-magma is idempotent or unital, and $\cdot$ is commutative then ${\sim}x = -x$.*

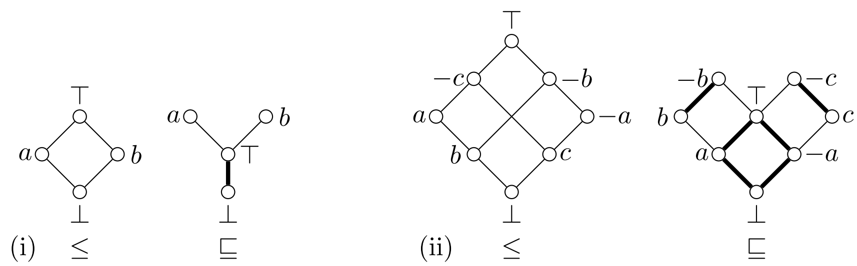

Figure 1: (i) The smallest iℓ-semilattice that is not unital, and (ii) the smallest ipo-semilattice that is not lattice-ordered. Note that $-\top = \bot$.

In a commutative idempotent ipo-semigroup there is another semilattice order, called the *multiplicative order*, that is defined by $x \sqsubseteq y \iff xy = x$, and these po-algebras are called *ipo-semilattices*. The most prominent examples of ipo-semilattices are Boolean algebras $(A, \cdot, +, -)$, where $x + y = -(-x \cdot -y)$. They arise as the case when the partial order $\leq$ and the semilattice order $\sqsubseteq$ coincide. More generally, ipo-semilattices are determined by the two relations $\leq$, $\sqsubseteq$ and the involution $-$ (see e.g. Figure 1).

Note that an element t in an ipo-semilattice is the multiplicative identity if and only if t is the top element in the multiplicative order, hence an ipo-semilattice is unital if and only if the multiplicative order has a top element.

We give a description of the structure of finite iℓ-semilattices, and provide some partial structural results for finite ipo-semilattices. For finite commutative idempotent involutive residuated lattices (i.e. unital iℓ-semilattices) a structural description has been given in [4]. We present a more global approach for ipo-semilattices based on Płonka sums of Boolean algebras. Similar methods have been used in [3] to describe the structure of odd and of even involutive FL_e-chains. Inspired by the duality for involutive bisemilattices [2], we also give a more compact dual description of finite ipo-semilattices based on semilattice direct systems of partial maps between sets. We present an algorithm to construct finite ipo-semilattices from their dual description, and an algorithm to enumerate the dual objects up to isomorphism. Some of our investigations were aided by computations using Prover9 and Mace4 [5].

References

[1] A. M. W. Glass: Partially ordered groups. World Scientific, 1999.

[2] Bonzio, S., Loi, A., Peruzzi, L.: A duality for involutive bisemilattices, Studia Logica, (2019) 107: 423–444, https://doi.org/10.1007/s11225-018-9801-0.

[3] Jenei, S.: Group-representation for even and odd involutive commutative residuated chains. Studia Logica, 2022, https://doi.org/10.1007/s11225-021-09981-y.

[4] P. Jipsen, O. Tuyt and D. Valota: *The structure of finite commutative idempotent involutive residuated lattices*, Algebra Univers. 82, 57, 2021, https://doi.org/10.1007/s00012-021-00751-4.

[5] W. McCune: Prover9 and Mace4. http://www.cs.unm.edu/~mccune/prover9/, 2005–2010.

[6] D. Pigozzi: Partially ordered varieties and quasivarieties. Unpublished lecture notes at https://orion.math.iastate.edu/dpigozzi/notes/santiago_notes.pdf, 2004, 1–26.

[7] J. Płonka: *On a method of construction of abstract algebras*, Fundamenta Mathematicae 61(2) 1967, 183–189.