# First order doctrines as bipresentable 2-categories

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First order doctrines are 2-categories corresponding to different fragments of first order logic; their objects are small categories endowed with a certain structure allowing to see them as syntactic categories for first order theories, where one can interpret connectors and inference rules. Prominent examples of doctrines are **Lex**, for *left exact* categories, corresponding to cartesian logic; **Reg**, for *regular* categories, corresponding to regular logic; **Coh** for *coherent categories*, corresponding to coherent logic; but also the 2-categories **Ext** of *extensive* categories, **Adh** of *adhesive* categories, **Ex** of *exact* categories, $\mathbf{Pretop}_\omega$ of *finitary pretopoi*, or **BoolPretop** of *boolean* finitary pretopoi.

Those doctrines can also be seen as higher-dimensional versions of the different categories of propositional algebras, as $\wedge - \mathbf{Slat}$, the category of meet-semilattices, **DLat**, the category of bounded distributive lattices, **Bool** the category of boolean algebras and so on... A common feature of most of those categories of propositional algebras is that they are *finitely presentable*: they are cocomplete and generated under filtered colimits by an essentially small subcategory of compact objects, which means in some sense that arbitrary objects can be constructed from simpler ones in a nice way. Finitely presentable categories are known to enjoy a lot of excellent properties and provide a framework generalizing universal algebra, a reason for which the 1-categorical theory of presentability, as well as the more general theory of *accessibility*, have become classical topics at the intersection of category theory and model theory since [8] and [1].

The purpose of this talk, which will be based on [4], is to prove the first order doctrines aforementioned to be themselves finitely presentable in a convenient 2-dimensional sense. Previous proposal for 2-dimensional accessibility and presentability can be found in [6] and [2] in the stricter context of enriched categories. However, capturing first order doctrines as examples requires a more relaxed version involving weaker notion of filteredness and colimits: for instance, **Lex** is not 2-presentable in the sense of [6] because it has only bicolimits and not all strict ones, beside issues about its expected rank of 2-accessibility in the sense of [2].

To fix this, we introduce here relaxed notions of *bi-accessible* and *bipresentable* 2-categories and connect them to the recent advance of [3] on the theory of *flat pseudofunctors*. Our notion relies on [7] notion of *bifilteredness*, together with a convenient notion of *bicompact objects* enjoying the analog property of compact objects, against bifiltered bicolimits. We then define finitely bi-accessible categories as those having bifiltered bicolimits and an essentially small subcategory of bicompact objects generating them under bifiltered bicolimits; finitely bipresentable 2-categories are as those that are moreover bicocomplete - but similarly to the one dimensional case, this amounts to having weighted pseudolimits. We then prove that categories of flat pseudofunctors are bi-accessible - and bipresentable if their domain admits finite weighted bilimits,

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the latter result being part of a categorification of the well known *Gabriel-Ulmer duality*, exhibiting in some sense finitely bipresentable 2-categories as 2-categories of models of "finite bilimit 2-sketches".

Finally, we prove that the 2-category of pseudo-algebras and pseudomorphisms for a finitary pseudomonad on a finitely bipresentable 2-category is itself finitely bipresentable. This captures in particular the example of **Lex**, for its bifiltered bicolimits can be shown to be computed in **Cat**. Then, invoking the powerful paradigm of *lex colimits* introduced by [5], we prove that, for a class of finite weights $\Phi$, the corresponding 2-category of $\Phi$-*exact* categories is finitely bipresentable: but this captures all the remaining doctrines defined from exactness properties as **Reg**, **Ex**, **Coh**, **Adh**, **Ext** and $\mathbf{Pretop}_\omega$.

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