Regular algebras over semimonads

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1 Semimonads

A monoid structure on a set M induces a monad structure on the endofunctor $- \times M$ of Set. Algebras over this monad are sets carrying a right M-action, meaning sets A along with a map

$$A \times M \to A$$

satisfying $(am)m' = a(mm')$ and $a1 = a$. More generally, this situation occurs whenever M is a monoid in a monoidal category, giving us the monad $- \otimes M$.

We can go through the same process starting with a semigroup S in a monoidal category, in which case we get a semimonad $- \otimes S$, a **semimonad** meaning a monad without a unit η. Even if S is made into a monoid by some unit, the category of algebras over the semimonad $- \otimes S$ is in non-trivial cases strictly larger than the category of algebras over the monad $- \otimes S$, since the latter need to be compatible with the unit ($a1 = a$ in the case of a monoid in Set). There is, however, an easy condition which characterises semimonad algebras that are compatible with the unit.

Proposition 1.1. *Suppose (T, μ) is a semimonad on a category $\mathcal{C}$ and $\xi\colon TA \to A$ is a T-algebra. If there exists an $\eta\colon 1 \to T$ such that (T, μ, η) is a monad, then the following statements are equivalent:*

1. *$\xi\colon TA \to A$ is an epimorphism,*
2. *$\xi\colon TA \to A$ is a split epimorphism,*
3. *the following is a coequalizer diagram*

$$\begin{tikzcd} TTA \arrow[r, shift left, "\mu_A"] \arrow[r, shift right, "T(\xi)"'] & TA \arrow[r, "\xi"] & A, \end{tikzcd}$$

4. *$\xi\colon TA \to A$ is an algebra over the monad (T, μ, η).*

2 Regular algebras

Consider an arbitrary semimonad (T, μ) on $\mathcal{C}$. In this case the conditions of Proposition 1.1 need not be equivalent. What kind of algebras over (T, μ) should we consider? We could use the category of all algebras over a semimonad, or we could consider the category of algebras subject to one of the conditions in Proposition 1.1.

While the properties of the category of all algebras are the easiest to describe, it is known from the $T = -\otimes S$ case that there are several properties which transfer poorly from monoids to semigroups without restricting the category of algebras to a suitable subcategory. For example, the only autoequivalences of the category of all algebras over a semimonad of the form $- \times S$ are the ones isomorphic to the identity functor, while in the monad case the autoequivalences can be much more numerous.

The category of algebras satisfying condition (3) of Proposition 1.1 appears to be a good choice. Such algebras have been studied by various authors in the $-\otimes S$ case for different choices of monoidal category, often under the name **firm** algebras (such as in [1]), although following [2], we will call such algebras **regular**. A large part of our results come from translating results about $- \otimes S$ into the general semimonad setting. This is quite straightforward in the situation that we will describe next.

3 Adjoint semimonads

If we want to prove things about the category of regular algebras of a semimonad T, it would be very helpful if T preserved coequalizers. In the case of semimonads of the form $- \otimes S$, this condition is naturally satisfied by assuming that the ambient monoidal category is closed. Furthermore, in that case the semimonad $- \otimes S$ will have right adjoint.

Motivated by this, we consider adjoint semimonads L, meaning semimonads (L, μ) such that L has a right adjoint R. It is well known that the right adjoint R of a semimonad will carry a cosemimonad structure and that the category of coalgebras over R will be isomorphic to the category of algebras over L via the correspondence

$$L(A) \to A \quad \Leftrightarrow \quad A \to R(A)\,.$$

However, this isomorphism between algebras and coalgebras need not be compatible with the notion of regularity. A coalgebra corresponding to a regular algebra need not satisfy the dual of the regularity condition. However, if it does, we say that the algebra (and the corresponding coalgebra) is **coregular**.

This gives us another class of algebras to consider, but for certain semimonads the category of regular algebras is equivalent to the category of coregular algebras. For example, we can achieve this by assuming things about the coreflection functor of algebras into regular algebras, such as the coreflector acting by taking the coequalizer of the pair

$$LLA \underset{L(\xi)}{\overset{\mu_A}{\rightrightarrows}} LA\,.$$

Finally, in this context, the epimorphism conditions of Proposition 1.1 are of renewed interest, since under suitable assumptions the category of regular algebras is equivalent to the category of algebras such that $L(A) \to A$ is a nice epimorphism and the corresponding $A \to R(A)$ is a nice monomorphism. This assumption on algebras is simple and self-dual, which makes it an appealing alternative to regularity.

References

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