Dualities from categorical first principles

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Many dualities of interest relate a category of algebras with some other, non-algebraic category: often a category of spaces. Classical Stone duality is a quintessential example of this phenomenon, describing the dual equivalence of the category of Boolean algebras with the category of Stone spaces. There are a multitude of Stone-type dualities which vary on this theme. A notable example is *Priestley duality*, the dual equivalence of the category of distributive lattices with that of Priestley spaces.

In this talk, we present our work developing a categorical framework for such dualities. Our goal is a categorically elegant approach, unifying and simplifying their construction. We motivate and describe our framework by showing how classical Stone duality and Priestley duality may be derived by way of the *ultrafilter* and *prime filter monads* respectively. Monads play a central role in the category-theoretic formulation of general (universal) algebra: in this context, they may be regarded as a generalisation of algebraic closure operators, allowing infinitary operations and arbitrary underlying objects replacing sets. In the late 1960s, Manes showed that the category of algebras for the ultrafilter monad is equivalent to the category of compact Hausdorff spaces [1]. In 1997, Flagg proved an analogous result: that the ordered counterpart of compact Hausdorff spaces, i.e. *compact pospaces*, are algebras for the prime filter monad — a monad on the category of partially ordered sets [2].

These monads are induced by adjoint functors of a particular type, and there are canonical comparison functors (contravariant) between the categories of Boolean algebras and compact Hausdorff spaces (in the case of the ultrafilter monad) and distributive lattices and compact pospaces (in the case of the prime filter monad). We rely on a key fact: the comparison functors for the ultrafilter and prime filter monads are contravariant fully faithful functors, and restricting to the essential image of such functors will always yield a duality. Here, we recover Stone and Priestley dualities respectively.

We will proceed to discuss current work refining our framework. This involves characterising conditions under which the comparison functor is full and faithful, with an eye to reconciling with Birkhoff's subdirect representation theorem.

References

[1] Manes, E. A triple theoretic construction of compact algebras. In *Sem. on Triples and Categorical Homology Theory (ETH, Zurich, 1966/67)*. Springer, 80 (1969) 91–118

[2] Flagg, B. Algebraic theories of compact pospaces. *Topology and its Applications*. 77 (1997) 277–290