Probability via logic: semantic analysis and proof theory

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Providing good proof systems for probabilistic logics is a long standing problem in proof theory and logics for uncertainty. In 1990, [5] introduces a logic to reason about probabilities and its Hilbert style calculus that contains three types of axioms and rules: the ones that govern the arithmetical part, i.e., the reasoning about inequalities; the ones that axiomatise probabilities; and the rules and axioms of classical propositional logic. The proposed calculus has the advantage of being quite intuitive and easy to use, however, its axiomatisation is infinite. In 2020, [1] utilises a two-layered modal logic to formalise reasoning about probabilities. The proposed calculus consists of three parts: the rules and axioms of the logic of events (i.e. classical logic) or 'inner logic'; the 'outer logic' that formalises reasoning with probabilities; and finally, the modalities that transform events into probabilistic statements.

This work is a part of larger research project aimed at providing good proof systems for probabilistic logics and other logics of uncertainty in a uniform and modular way. In this project, we use a generalization of display calculi introduced by Belnap [2]. This choice is motivated by the following two reasons. Firstly, display calculi are by design modular, insofar they implement a neat division of labour between logical rules (introducing the connectives and relying on their minimal order-theoretic properties) and so-called structural rules (capturing the specific features of the logic under consideration). Secondly, they provides a framework in which cut-elimination, a crucial property of proof systems, can be proved in a principled way as an application of a general meta-theorem.

The main difficulties in applying the theory of display calculi to the probability logics lies in the handling of the operators like $+$ and $-$ (i.e. the (truncated) sum and difference, respectively) and their interaction with the probability operator P.

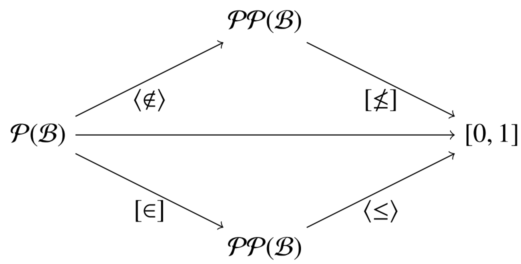
A specific and preliminary difficulty in the handling of probability operator is that it is a non-normal operator (with usual definition of join and meet on interval $[0, 1]$), so standard techniques and design choices are immediately banned. To overcome this issue, we use a generalization of the standard theory to a so-called multi-type environment (see, for instance, [7] as a prototypical examples of this approach). Here we consider an equivalent multi-type presentation of the algebraic semantics for probability logic, thanks to which we can define an appropriate formal translations of the original language into a new multi-type language (preserving validity and derivability of formulas).

A similar problem arises when considering the peculiar axiom of probability logic (involving the sum, the difference and the probability operator as well) given that it not **analytic inductive** [6] in the original language, so, once again, standard techniques cannot be applied.

In an ongoing work, we introduce a generalization of standard display calculi we used to capture Łucasewicz logic and, in particular, to deal with the peculiar axiom of the logic involving the sum (or, equivalently, the subtraction depending on the presentation of the axiom). This axiom is key to capture the interaction with the probability operator, and the specific design choices implemented in this calculus (motivated by an algebraic analysis of Łucasewicz logic) are imported here as well. Moreover, we use a specialized version of the algorithm ALBA to automatically generate the analytic structural rules equivalently capturing the axiom. Showing that such rule preserve the analyticity of the basic calculus is work in progress.

Below we expand on the treatment of the probability operator. The key idea is that the non-normal operators (like the conditional binary operator of conditional logics or the monotone unary modalities in non-normal modal logics) can be decomposed into the composition of normal modal operators [4]. In this work, we use a similar approach to deal with the probability operator P.

Let $\mathcal{B}$ be any set and $\mathcal{P}(\mathcal{B})$ be its power-set. Let $P : \mathcal{P}(\mathcal{B}) \to [0, 1]$ be a probability function on it.



Let $R_\in, R_\notin \subseteq \mathcal{P}(\mathcal{B}) \times \mathcal{B}$ be defined as follows. For any $a \in \mathcal{B}$, $A \in \mathcal{P}(\mathcal{B})$,

$$AR_\in a \quad \text{iff} \quad a \in A \quad \text{and} \quad AR_\notin a \quad \text{iff} \quad a \notin A.$$

Let $R_\leq, R_\nleq \subseteq [0,1] \times \mathcal{P}(\mathcal{B})$ be defined as follows. For any $\alpha \in [0,1]$, $A \in \mathcal{P}(\mathcal{B})$,

$$\alpha R_\leq A \quad \text{iff} \quad \alpha \leq P(A) \quad \text{and} \quad \alpha R_\nleq A \quad \text{iff} \quad \alpha \nleq P(A).$$

Let $A \subseteq \mathcal{B}$, and $U \subseteq \mathcal{P}(\mathcal{B})$ be any subsets of $\mathcal{B}$ and $\mathcal{P}(\mathcal{B})$ respectively. Let $[\in](A) = [R_\in](A)$, $\langle\notin\rangle(A) = \langle R_\notin\rangle(A)$, $\langle\leq\rangle(U) = \langle R_\leq\rangle(U)$, and $[\nleq](U) = [R_\nleq](U)$, where for any relation R, $[R]$ and $\langle R\rangle$ denote the box and diamond operators corresponding to the relation R on the given frame. Then, we have

Lemma 1. *For any $A \subseteq \mathcal{B}$, and $U \subseteq \mathcal{P}(\mathcal{B})$ we have (1) $[\in](A) = A^\downarrow$, (2) $\langle\notin\rangle(A) = (A^\uparrow)^c$, (3) $\langle\leq\rangle(U) = [0, \max\{P(A) \mid A \in U\}]$, and (4) $[\nleq](U) = [0, \min\{P(A) \mid A \in U^c\}]$.*

The following corollary follows immediately from the Lemma.

Corollary 2. *For any $A \subseteq \mathcal{B}$, $P(A) = \max(\langle\leq\rangle[\in](A)) = \max([\nleq]\langle\notin\rangle(A))$.*

Thus, under the identification of an interval with its largest element above, the corollary shows that the probability operator P can be decomposed into the combination of normal operators $\langle\leq\rangle$, $[\in]$, $[\nleq]$, and $\langle\notin\rangle$ in two ways. This decomposition allows us to write the probability axioms in the language of Łucasewicz logic expanded with the above modal operators. Therefore, the axioms of probability logic can be expressed in the above multi-type normal modal logic. We believe these techniques would allow us to introduce display-like calculi for probability logics and other (non-classical) logics of uncertainty such as the logics for probabilities and belief functions over Belnap-Dunn logic introduced in [3].

References

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