Modality in worlds with different logics

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Most non-classical logics can be understood in a modal structure with classical worlds (e.g. Priest's approach in [1]). Whats more, we can also make use of the Kripke style semantics to study modality independently of the assumption of a background logic – i.e. the logic regulating the non-modal connectives. However little is said about modal structures in which there are worlds operating in different logics (see. [2]). An important concern is whether or not it makes sense to evaluate the possibility of a statement in a world w by inspecting its truth value in a world w' in a alternative logic. As argued by Kripke in [3], when we are navigating through worlds, the referents for names should be fixed in order to make sense of intuitive distinctions between counterfactuals (e.g. "Newton could have died young" and "the inventor of calculus could have died young"). Likewise, we argue that something must be preserved when we change the logical background – for the validity of a formula can have different meanings when evaluated in different logics.

We address this issue by introducing a modal notion and structure that accommodate communication between logic systems by fixing a common lattice L where different logics build their semantics (see. [4]). We suggest that from a collection of logics with complete lattice semantics Σ, one should build a common lattice L (which always exist) that has Σ as a collection of complete sublattices. The common order offered in L can then be taken as the background where the appropriate communication of logical values occur. Necessity and possibility of a statement will not solely rely on the satisfaction relation in each world and the accessibility relation. Instead, the value of a formula $\Box\varphi$ will be defined in terms of a comparison between the values of φ in accessible worlds and the common lattice L. This is done by relativizing each value of φ in an accessed world w' to a value in the current world w using the **down-interpretation** or the **up-interpretation**:

Definition 1. *In a base lattice L, a value $a \in L$ is interpreted in a sublattice L' as:*

1. ***Down-interpretation*** *- the least value in L' that is larger than* ***all values of*** *L'* ***that are smaller than*** *a – formally, $a^{L'} = \bigcup_{L'}\{x \in L' \mid x \leq a\}$.*
2. ***Up-interpretation*** *- the largest value in L' that is smaller than* ***all values of*** *L'* ***that are bigger than*** *a – formally, $a^{L'} = \bigcap_{L'}\{x \in L' \mid x \geq a\}$.*

With this natural interpretation of multi-logic modal scenarios, we will show a series of cases where a formula φ can be said to be necessary/possible even though an/all accessible world/s falsify φ. This possibility arises from natural algebraic properties of sublattices.

Subsequently, we will characterize these modal structures by establishing conditions that imply validity of the axiom K and/or the rule of necessitation. We shall present this analysis in a limited setting, where we fix a unary function $(-)$ over the common lattice L for negation

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in the common lattice and interpret disjunction and conjunction as the joint $(+)$ and meet operations $(.)$. With implication $(x \to y)$ defined as $(-x + y)$ and a set of designated values $F \subseteq L$ defining validity, we obtain general conditions for a L'-world w to satisfy axiom K. We also define a semantical notion of necessitation restricted to worlds of a given sublattice L^*, showing the conditions over L^* and F required for this necessitation to hold.

We should further characterize the many logic modal structures with respect to the finite model property (FMP). This is an important property of modal systems and it is intimately related to decidability of modal logics. We say that a semantic has the FMP if, for every non valid formula, there is a structure with finite number of worlds that falsify the formula. Lattices being of varying complexities can produce scenarios where FMP fails. We will nevertheless show interesting classes of infinite lattices that still produce structures that have FMP.

Expanding the universe of modal structures to varying logics operating in each worlds opens up the possibility of investigating more nuanced notions of frames. Traditional modal theory define frames solely with respect to the accessibility relation between worlds. Now, not only we can say that the accessibility relation is for instance transitive, but that it also relates worlds with some lattices. Defining a relation of 'more classical' between sublattice of an L, we will produce frames where the accessibility relation goes from less classical worlds to more classical worlds. We will show some examples of this phenomena where the base lattice is produced by the twist of boolean algebras as proposed by Fidel in [5] and Vakarelov in [6].

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