Advantages and challenges posed by PNmatrices*

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Partial non-deterministic matrices (PNmatrices) are algebraic-like structures that were introduced in the beginning of this century [1, 2, 4] as a generalization of logical matrices, by allowing the connectives to be functionally interpreted as partial multi-functions, rather than functions. Herein, we survey some recent results showing the advantages brought by taking PNmatrices, instead of logical matrices, as primary semantic structures, and also discuss the challenges raised by such a generalization.

PNmatrices allow for finite characterizations of a much wider class of logics and general recipes for various problems in logic, such as procedures to constructively update semantics when imposing new axioms [8, 7], or effectively combining semantics for two logics, capturing the effect of joining their axiomatizations [6, 13]. Whenever the underlying language is expressive enough, PNmatrices also allow for general techniques for effectively producing analytic calculi for the induced logics, over which a series of reasoning activities in a purely symbolic fashion can be performed, including proof-search and countermodel generation [16, 5, 14].

Although logics of finite PNmatrices are still decidable and in coNP, recently, it was shown that several relevant problem known to be decidable for finite matrices become undecidable due to the incorporation of non-determinism (and partiality). Namely, given finite PNmatrices, the problems of checking if the induced logic has theorems, checking if the induced logics have the same set of theorems, or checking if the induced logics (as consequence relations) are the same are no longer decidable [9, 15].

In future research, we aim at a deeper understanding of PNmatrices, and their behaviour with respect to homomorphisms (actually, strict homomorphisms that correspond to so-called rexpansions [3]), congruences, and other basic operations and relations, extending the scope of Abstract Algebraic Logic results concerning logical matrices [10]. These are challenging questions. Gräzer [11] has shown that every multi-algebra can be obtained as a quotient of an algebra by an equivalence relation. Still, a quotient of a PNmatrix by an equivalence relation respecting its filter is still a PNmatrix, but may be defining a weaker logic. Several alternative generalizations of the traditional Leibniz operator can be explored, but it seems difficult to obtain a reasonable notion of reduced PNmatrix that may allow for a Lindenbaum-like construction based on PNmatrices. It is also unclear, even for finite matrices, how to prove (or disprove) whether a given quotient defines the same logic. These difficulties seem to be connected with the unfamiliar behaviour of abbreviations in the presence of non-determinism, as defined connectives lose their relationship with the primitive connectives with which they were defined [12], and which complicates the association of (fragments of) logics with certain clones of multi-functions.

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