Pointfree topology and Constructive Mathematics

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Remember from yesterday: frames

Opens can be understood as verifiable properties.

Verifiable properties can be studied abstractly as elements of algebras called frames.

Definition

A frame is a poset with finite meets and arbitrary joins satisfying the distributivity condition $a \wedge \bigvee_\alpha b_\alpha = \bigvee_\alpha a \wedge b_\alpha$.

The lattice of opens of any topological space gives a frame.

Points of a frame are given by frame homomorphisms to the frame of truth values $\Omega = \mathcal{P}(1)$. These are models of the geometric theory represented by the frame.

Remember from yesterday: presentations

An advantage of the pointfree approach is that frames can be presented by generators and relations. Presentations can also be understood as axiomatising a geometric theory.

For example, the locale reals of is presented by

$$
\begin{aligned}
\mathcal{O}\mathbb{R} = \big\langle (\!(q,\infty)\!), (\!(-\infty, q)\!),\ q \in \mathbb{Q} \mid \\
&(\!(p,\infty)\!) = \bigvee_{q>p} (\!(q,\infty)\!),\ (\!(-\infty,q)\!) = \bigvee_{p<q} (\!(-\infty,p)\!), \\
&\bigvee_{q\in\mathbb{Q}} (\!(q,\infty)\!) = 1,\ \bigvee_{q\in\mathbb{Q}} (\!(-\infty,q)\!) = 1, \\
&(\!(-\infty,q)\!) \wedge (\!(p,\infty)\!) = 0 \ \text{ for } p \geq q, \\
&1 \leq (\!(p,\infty)\!) \vee (\!(-\infty,q)\!) \ \text{ for } p < q \big\rangle.
\end{aligned}
$$

This is obtained from the geometric theory of Dedekind cuts on $\mathbb{Q}$.

Example — Cantor space

The points of Cantor space $2^{\mathbb{N}}$ are infinite sequences of bits 0 or 1.

We can verify if the n^{th} element of the sequence is 0 and 1, giving generators z_n and u_n.

This suggests $\mathcal{O}(2^{\mathbb{N}}) \cong \langle z_n, u_n,\ n \in \mathbb{N} \mid z_n \wedge u_n = 0,\ z_n \vee u_n = 1 \rangle$.

The points correspond to the decidable subsets of $\mathbb{N}$ as we expect.

Example — Sierpiński space

Consider the frame $\langle g \rangle$ with one generator and no relations.

Points correspond to truth values.

This is the frame of opens of Sierpiński space: the set of points is Ω and the topology is generated by the single subbasic open $\{\top\}$.

More generally, the points of the free frame on G generators will be given by subsets of G. The space is homeomorphic to $\mathbb{S}^G$.

Example — the Stone spectrum of a distributive lattice

Let L be a bounded distributive lattice. The Stone spectrum of L is the space of prime filters of L.

A prime filter is a subset $F \subseteq L$ such that

- if $a \leq b$ and $a \in F$ then $b \in F$,
- $1 \in F$,
- if $a \in F$ and $b \in F$ then $a \wedge b \in F$,
- $0 \notin F$,
- if $a \vee b \in F$ then $a \in F$ or $b \in F$.

This gives the presentation

$$\langle \overline{a} \text{ for } a \in L \mid \overline{1} = 1, \overline{a} \wedge \overline{b} = \overline{a \wedge b}, \overline{0} = 0, \overline{a} \vee \overline{b} = \overline{a \vee b} \rangle.$$

where $\overline{a}$ is a basic proposition asserting that a lies in the filter.

Example — surjections from $\mathbb{N}$ to X

Fix a set X and consider the following geometric theory.

Basic propositions are denoted by $[f(n) = x]$ for $n \in \mathbb{N}$ and $x \in X$.

- $[f(n) = x] \wedge [f(n) = y] \vdash [\![x = y]\!]$ for $x, y \in X$,
- $\top \vdash \bigvee_{x \in X} [f(n) = x]$ for $n \in \mathbb{N}$,
- $\top \vdash \bigvee_{n \in \mathbb{N}} [f(n) = x]$ for $x \in X$.

The points correspond to surjections from $\mathbb{N}$ to X.

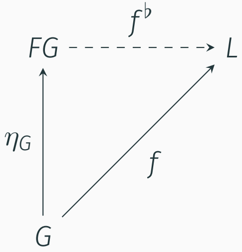
If X is chosen to be large enough there are no such surjections!

However, it is still a nontrivial locale.

Free frames

Free frames

Presentations first require us to construct free frames.



The forgetful functor from $\mathbf{Frm}$ to $\mathbf{Set}$ factors through $\wedge$-$\mathbf{SLat}$.

So to find the free frame on the set G we can first find the free $\wedge$-semilattice on G and then take the free frame on that.

Free semilattices

The free semilattice on G can be found as a quotient of the free monoid on G.

But it will be helpful to have a simpler description.

Classically, the free semilattice on G is given by $\mathcal{P}_{\mathrm{fin}}(G)$, the set of finite subsets of G with the operation of union.

Finiteness can be a little subtle constructively.

Finiteness

Natural numbers behave like they do classically, since they have decidable equality: $\forall n, m \in \mathbb{N}.\ n = m \vee n \neq m$.

Definition

A set X is Kuratowski-finite is if there is a natural number n and a *surjection* $e \colon [n] \twoheadrightarrow X$, where $[n] := \{i \in \mathbb{N} \mid i < n\}$.

If X is Kuratowski-finite then either $X = \emptyset$ or $\exists x \in X$.
In the latter case we say X is inhabited.

Subsets of finite sets are *not* necessarily finite!
If $\{* \mid p\}$ is Kuratowski-finite then $p \vee \neg p$!

However, Kuratowski-finite sets are closed under images and Kuratowski-finite unions.

Lemma

Let L be a $\vee$-semilattice and let $S \subseteq L$ be a Kuratowski-finite subset. Then $\bigvee S$ exists and is equal to $\bigvee_{i=0}^{n-1} e(i)$ for any surjection $e\colon [n] \twoheadrightarrow S$.

Proof.

For each $s \in S$ there is an $j \in [n]$ such that $s = e(j) \leq \bigvee_{i=0}^{n-1} e(i)$ and hence $\bigvee_{i=0}^{n-1} e(i)$ is an upper bound of S.

Now suppose $u \in L$ is an upper bound of S. Then for each $j \in [n]$, we have $e(j) \leq u$ and hence $\bigvee_{i=0}^{n-1} e(i) \leq u$. Thus, this is the least upper bound of S. $\square$

Free semilattices, again

Proposition

The set $\mathcal{P}_{\mathrm{fin}}(G)$ of Kuratowski-finite subsets of G (under union) is the free $\vee$-semilattice on G.

Proof.

Certainly, $\mathcal{P}_{\mathrm{fin}}(G)$ is a $\vee$-semilattice. Now let L be a $\vee$-semilattice and consider a function $f\colon G \to L$.

We define $f^\flat(S) = \bigvee_{s \in S} f(s)$, which exists by the lemma. Moreover, this definition is forced by $f^\flat(\{s\}) = f(s)$.

It remains to show $f^\flat$ preserves 0 and $\vee$, but this easy to check. $\square$

Of course, this means the free $\wedge$-semilattice on G is $\mathcal{P}_{\mathrm{fin}}(G)^{\mathrm{op}}$.

Free frames, again

Recall that a downset in a poset P is a subset $S \subseteq P$ such that if $s \leq t$ and $t \in S$ then $s \in S$. We set $\downarrow a = \{b \in P \mid b \leq a\}$.
The downsets on a poset form a frame.

Proposition

The free frame the $\wedge$-semilattice M is the frame of downsets $\mathcal{D}M$.

Proof.
Let L be a frame and suppose $f\colon M \to L$ is a $\wedge$-semilattice homomorphism. We must show there is a unique frame homomorphism $f^\flat\colon \mathcal{D}M \to L$ such that $f^\flat(\downarrow m) = f(m)$.

By join preservation we need $f^\flat(D) = f^\flat(\bigcup_{d \in D} \downarrow d) = \bigvee_{d \in D} f(d)$.

It is easy to see this preserves joins and 1. For $D, E \in \mathcal{D}M$, we have $f^\flat(D) \wedge f^\flat(E) = \bigvee_{d \in D, e \in E} f(d) \wedge f(e) \leq \bigvee_{a \in D \cap E} f(a) \leq f^\flat(D \cap E)$. □

Frame presentations

Thus, $\mathcal{D}(\mathcal{P}_{\mathrm{fin}}(G)^{\mathrm{op}})$ is the free frame on G!

Now consider a presentation $\langle G \mid R \rangle$ where R represents a set of formal equalities or inequalities in the free frame on G.

An inequality $x \leq y$ in R can be rewritten as an equality $x = x \wedge y$ or $y = x \vee y$, so we may restrict our attention to equalities.

The presented frame is obtained by quotienting the free frame on G so that the specified equations from R hold in the quotient.

Sublocales and frame quotients

Congruences

As for any algebraic structure, frame quotients can be described by congruences.

Definition

A congruence on a frame L is an equivalence relation on L that is also a subframe of $L \times L$.

The quotient by such an equivalence relation is a frame quotient.

The congruences on L are closed under intersections and thus form a complete lattice.

The least congruence containing a set of pairs $R \subseteq L \times L$ is called the congruence generated by R and written $\langle R \rangle$.

Then $\langle G \mid R \rangle = \langle G \rangle / \langle R \rangle$.

Sublocales

What is the spatial intuition for frame quotients?

- Since frames are dual to spaces, quotients should correspond to some notion of subspace.
- Adding axioms to a geometric theory reduces the number of models (with basic propositions / opens coming from the parent theory).
- If $S \subseteq X$ then the map $U \mapsto U \cap S$ restricting opens to S is a surjective frame map.

Thus, we think of frame quotients as corresponding to sublocales.

Open sublocales

We expect an open $a \in \mathcal{O}X$ to give an open sublocale of X.

The frame of this sublocale is simply $\downarrow a \subseteq \mathcal{O}X$.

The quotient map sends $u \in \mathcal{O}X$ to $u \wedge a$.

The corresponding congruence is $\Delta_a = \{(u, v) \mid u \wedge a = v \wedge a\}$.

Lemma

The open congruence $\Delta_a = \langle(a, 1)\rangle$.

Proof.
Certainly, $(a, 1) \in \Delta_a$.

Now suppose $u \wedge a = v \wedge a$. In $\langle(a, 1)\rangle$, we have $u \sim u$ and $a \sim 1$, so $u \wedge a \sim u \wedge 1 = u$. Similarly, $v \wedge a \sim v$. Thus, $u \sim v$. □

Closed sublocales

Every element $a \in \mathcal{O}X$ also induces a closed sublocale.

The congruence associated to such a sublocale is $\nabla_a = \langle(0, a)\rangle$.

Similarly to before, we can prove $\nabla_a = \{(u, v) \mid u \vee a = v \vee a\}$.

The quotient is isomorphic to $\uparrow a$ with quotient map $u \mapsto u \vee a$.

Lemma

The open and closed sublocales induced by a are mutual complements in the lattice of sublocales.

Proof.

Firstly, $\nabla_a \vee \Delta_a = \langle(0, a)\rangle \vee \langle(a, 1)\rangle \supseteq \langle(0, 1)\rangle = L \times L$. Now take $(u, v) \in \nabla_a \cap \Delta_a$. Consider $(u \wedge v) \vee (u \wedge a) = u \wedge (v \vee a)$. Since $(u, v) \in \nabla_a$, we know $v \vee a = u \vee a$. So $u \wedge (v \vee a) = u$. Similarly, $(u \wedge v) \vee (v \wedge a) = v$. But also $u \wedge a = v \wedge a$ and so these agree. □

Open sublocales of discrete locales

A set X can be viewed as a space with the discrete topology.
The frame of opens of the space X is just the powerset Ω^X.

Lemma

Discrete spaces are sober — all points of Ω^X come from points of X.

Proof.

The points of Ω^X are filters $\mathcal{F}$ such that if $\bigcup \mathcal{S} \in \mathcal{F}$ then $\exists S \in \mathcal{S} \cap \mathcal{F}$.

But $\bigcup_{x \in X} \{x\} = X \in \mathcal{F}$ and so $\{x\} \in \mathcal{F}$ for some $x \in X$. Now suppose $U \in \mathcal{F}$. Then $U \cap \{x\} \in F$. But $U \cap \{x\} = \bigcup \{\{x\} \text{ (fixed)} \mid x \in U\} \in \mathcal{F}$. Thus, this set is inhabited and $x \in U$. So $\mathcal{F} = \{U \subseteq X \mid x \in U\}$. $\square$

Open sublocales of the discrete locale X are simply *subsets*.
Opens are, of course, subsets $S \subseteq X$ and $\downarrow S \cong \Omega^S$.

Closed sublocales of discrete locales

Since closed sublocales and open sublocales are complements, sublocales of discrete spaces which are both closed and open correspond to decidable subsets.

Closed sublocales can be specified by their complementary subsets. This explains some strange definitions in constructive algebra.

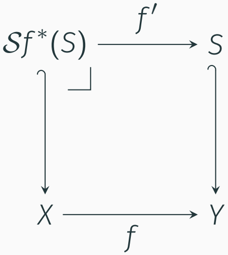
Definition

Let R be a ring. An anti-ideal of R is a subset $A \subseteq R$ such that

- $0 \notin A$,
- if $x + y \in A$ then $x \in A \vee y \in A$,
- If $xy \in A$ then $x \in A \wedge y \in A$.

Preimages of sublocales

We can easily define preimages of sublocales categorically.

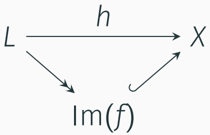


This makes the sublocale lattice into a functor $\mathcal{S}\colon \mathrm{Loc}^{\mathrm{op}} \to \mathrm{Pos}$.

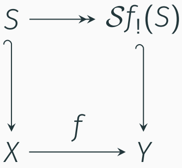
By standard algebraic arguments, if C_S is the congruence for S then the congruence corresponding to $\mathcal{S}f^*(S)$ is $\langle (f^* \times f^*)(C_S) \rangle$.

Note that preimages of open/closed sublocales are open/closed.

Like for any algebraic structure, frame homomorphisms have image factorisations.



Thus a locale map $f\colon X \to Y$ factorises into an epimorphism followed by a sublocale inclusion. We can use this to define images of sublocales.



We have $C_{\mathcal{S}f_!(S)} = (f^* \times f^*)^{-1}(C_S)$. The map $\mathcal{S}f_!$ is left adjoint to $\mathcal{S}f^*$.