Pointfree topology and Constructive Mathematics

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TACL Summer School
June 2022

An application to fibrewise topology

An example application

Let's end with a simple example, which shows the power this approach to topology.

We will prove a constructive version of the Extreme Value Theorem.

Classically this says that if X is compact, then every continuous function $f \colon X \to \mathbb{R}$ attains its maximum value.

Our proof will be based on that given in
Paul Taylor. "A lambda calculus for real analysis". In: *J. Log. Anal.* 2.5 (2010), pp. 1–115

Upper and lower reals

Classically, real numbers have all bounded, inhabited suprema and infima.

This is not so constructively. What should the supremum of $\{0\} \cup \{1 \mid p\}$ be?

Completing $\mathbb{Q}$ under inhabited suprema gives the lower reals.

These are constructed via one-sided Dedekind cuts using only lower sets L instead of pairs (L, U).

Completing under inhabited infima gives the upper reals, which are constructed using only upper sets U.

Upper and lower reals

For the lower reals we have a basic proposition ℓ_q for each $q \in \mathbb{Q}$ and the three axioms of Dedekind cuts that apply only to L:

- $\ell_q \vdash \ell_p$ for $p \leq q$
- $\ell_p \vdash \bigvee_{q>p} \ell_q$ for $p \in \mathbb{Q}$
- $\top \vdash \bigvee_{q \in \mathbb{Q}} \ell_q$

This gives the presentation

$$\mathcal{O}\overrightarrow{\mathbb{R}} = \langle \ell_q,\ q \in \mathbb{Q} \mid \ell_p = \bigvee_{q>p} \ell_q,\ \bigvee_{q \in \mathbb{Q}} \ell_q = 1 \rangle.$$

Similarly, for the upper reals we have

$$\mathcal{O}\overleftarrow{\mathbb{R}} = \langle u_q,\ q \in \mathbb{Q} \mid u_q = \bigvee_{p<q} u_p,\ \bigvee_{q \in \mathbb{Q}} u_q = 1 \rangle.$$

Suprema of overt locales

Let V be a positive overt sublocale of $\mathbb{R}$.

We can define the supremum of V as a lower real λ by $q < \lambda \iff \exists x \in V.\ q < x$.

It is fairly intuitive that this defines a lower real, but let us spell it out formally. This means $q \in \ell_q \iff V \between \ell_q$. From the properties of $\mathbb{R}$, we have $\ell_p = \bigvee_{q>p} \ell_q$. Now $V \between (-)$ is given by the composite of the frame quotient $\mathcal{O}R \twoheadrightarrow \mathcal{O}V$ and the left adjoint $\exists_V$ and hence preserves joins. So $V \between \ell_p \iff V \between \bigvee_{q>p} \ell_q$. Thus, the assignment of truth values given by λ satisfies the first relation of $\mathcal{O}\overrightarrow{\mathbb{R}}$. Now in $\mathcal{O}\mathbb{R}$ we have $\bigvee_{q \in \mathbb{Q}} \ell_q = 1$ and since V is positive we know $V \between 1$. Thus, we similarly have that λ satisfies the second relation and λ is indeed a lower real.

Suprema of compact locales

Let K be a compact sublocale of $\mathbb{R}$.

We can define the supremum of K as an *upper* real υ by
$\upsilon < q \iff \forall x \in K.\ x < q$.

This time we omit the details, but we use that by compactness
$[\![K \subseteq (-)]\!]$ preserves the directed joins $\bigvee_{p<q} u_p$ and $\bigvee_{q \in \mathbb{Q}} u_q$.

Suprema of compact overt locales

Let K be a sublocale of $\mathbb{R}$ that is positive, overt *and* compact.

We claim the prior definitions define the supremum as a Dedekind real ρ.

It just remains to check

1. if $p < \rho$ and $\rho < q$ then $p < q$,
2. if $p < q$ then either $p < \rho$ or $\rho < q$.

Suprema of compact overt locales

Suppose $p < \rho$ and $\rho < q$. By definition this means $\exists x \in K.\ p < x$ and $\forall x \in K.\ x < q$.

Intuitively, we should be able to argue $p < x < q$ and hence $p < q$.

This is justified formally by noting $K \between (\!(p, \infty)\!)$ and $K \subseteq (\!(-\infty, q)\!)$ implies $(\!(-\infty, q)\!) \between (\!(p, \infty)\!)$. But by this very axiom in $\mathcal{O}\mathbb{R}$ we have $(\!(-\infty, q)\!) \wedge (\!(p, \infty)\!) = 0$ for $q \leq p$ and hence $p < q$.

Now suppose $p < q$. We must prove that either $p < \rho$ or $\rho < q$.

By the definition of ρ we must show $\exists x \in K.\ p < x$ or $\forall x \in K.\ x < q$, or in other words $K \between (\!(p, \infty)\!)$ or $K \subseteq (\!(-\infty, q)\!)$.

By the similar axiom in $\mathbb{R}$ we have $(\!(-\infty, q)\!) \vee (\!(p, \infty)\!) = 1$ and so $K \subseteq (\!(-\infty, q)\!) \vee (\!(p, \infty)\!)$.

The result then follows from the following lemma.

Lemma

Let K be a compact overt locale and suppose $a \vee b = 1$ in $\mathcal{O}K$. Then either $b > 0$ or $a = 1$.

Proving the lemma

Lemma

Let K be a compact overt locale and suppose $a \vee b = 1$ in $\mathcal{O}K$. Then either $b > 0$ or $a = 1$.

Proof.

By overtness we have $b = \bigvee\{b \mid b > 0\}$. Then by compactness $1 = a \vee b = a \vee \bigvee\{b \mid b > 0\}$ has a Kuratowski-finite subcover.

Thus we have $c_1 \vee c_2 \vee \cdots \vee c_n = 1$ where for each i, either $c_i = a$ or $c_i = b > 0$. Since there are finitely many of these we can conclude that either $c_i = a$ for all i or $b > 0$.

In the former case $a = 1$ and in the latter case $b > 0$. □

The supremum is a maximum

Thus, the positive compact overt locale K has a Dedekind real ρ as its supremum. We now show it is a maximum.

Since $\mathbb{R}$ is Hausdorff and K is compact, K is a *closed* sublocale. To show $\rho \in K$ we must show ρ does not lie in its open complement.

Intuitively, ρ lies in this open if $\forall x \in K.\ x \neq \rho$. More formally, this says $K \subseteq \bigvee_{p<\rho} ((-\infty, p)) \vee \bigvee_{\rho<q} ((q, \infty))$. We must show this is *false*.

Suppose it is true. By compactness, $K \subseteq ((-\infty, p)) \vee ((q, \infty))$ for some $p < \rho$ and some $q > \rho$. By the lemma this implies that either $K \subseteq ((-\infty, p))$ or $K \between ((q, \infty))$.

Now recall that ρ is defined by $p < \rho \iff K \between ((p, \infty))$ and $\rho < q \iff K \subseteq ((-\infty, q))$. But since $((-\infty, p)) \wedge ((p, \infty)) = 0$ and $((q, \infty)) \wedge ((-\infty, q)) = 0$, these contradict both options above!

Now since the image of a compact overt positive locale under a locale map is compact, overt and positive, we have arrive at the theorem.

Theorem (Extreme Value Theorem)
If K is any compact overt positive locale and $f\colon K \to \mathbb{R}$, then f has a maximum value given by a Dedekind real number ρ.

Note that since $\mathbb{R}$ is Hausdorff, ρ is a closed sublocale of $\mathbb{R}$. So its preimage under f is a closed and hence compact sublocale of K.

This sublocale of K is positive since the sublocale $\{\rho\}$ is. (Though it does *not* necessary contain a point.)

Sheaf toposes

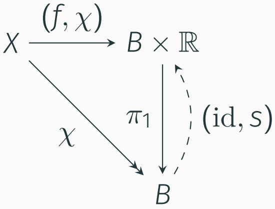
We can now interpret this result in a topos.
In particular, if B is a locale there is a topos $\mathsf{Sh}(B)$ of sheaves on B.

We can use the following dictionary.

Constructive concept	Interpretation in $\mathsf{Sh}(B)$
Locale	Locale map into B
Point	Section
Compact locale	Proper map
Overt locale	Open map
Positive compact locale	Proper surjection
Positive overt locale	Open surjection
$\mathbb{R}$	$\pi_1 \colon B \times \mathbb{R} \to B$

A consequence of our theorem

In this way, we can immediately obtain the following diagram.



Finally, translating from locales to spaces, we obtain the following.

Corollary

Let B be a T_D topological space, $\chi\colon X \to B$ a proper and open surjection and $f\colon X \to \mathbb{R}$ a continuous function. Then the function $s\colon B \to \mathbb{R}$ defined by $s(b) = \max_{x \in \chi^{-1}(b)} f(x)$ is continuous. Moreover, even if χ is not open, s is still upper semi-continuous.

Summary and further resources

What we have learnt

- Topology is the study of verifiable properties.
- Pointfree topology studies these properties abstractly as a lattice with arbitrary joins satisfying $a \wedge \bigvee_\alpha b_\alpha = \bigvee_\alpha a \wedge b_\alpha$.
- These frames can be presented by generators and relations.
- The presentations axiomatise geometric theories.
- The theory of Dedekind cuts gives the locale of reals!
- Hausdorff locales are those for which equality is refutable.
- In discrete locales, equality is always verifiable.
- We can universally quantify over compact locales.
- We can existentially quantify over overt locales.

What we have learnt

- Constructive proofs can be interpreted in many different toposes, yielding different results for free.
- Thinking topologically makes some of the surprising aspects of constructive mathematics more intuitive.
- With the pointfree approach we can almost always recover some version of the classical results constructively.
- This even includes versions of the excluded middle and the axiom of choice!

Useful references

- Steven Vickers. *Topology via logic*. Cambridge: Cambridge University Press, 1989.
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- Peter T Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*. Vol. 2. Oxford: Oxford University Press, 2002.