From Metric Spaces to Quantale-Enriched Categories

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Topology, Algebra, and Categories in Logic

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Learning goals for this lecture series

- Embrace (enriched) category theory as a guide for analytic inquiry
- Appreciate the quantalic structure of the real half-line as the key for studying metrics
- Get familiar with other important quantales and study the categories enriched in them
- Study the core of the theory: cocompleteness *vs* injectivity *vs* pseudo-algebraicity,
- in particular: Cauchy *vs* Lawvere
- Feel prepared to study monad-quantale-enriched categories (*Monoidal Topology*),
- normed/weighted categories, metrically enriched categories, metagories, *etc*.

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Lectures

1. Metrics: from Frechét via Hausdorff to Lawvere
2. Quantales and the (small) categories enriched in them
3. Distributors and the presheaf monad
4. Weighted colimits, tensors, conical infima
5. Pseudo-algebras of the presheaf monad, injectivity
6. Cauchy- and Lawvere-completeness
7. A glance at normed/weighted categories

1.1 Fréchet 1906

A Frechét metric $d : X \times X \longrightarrow \mathbb{R}$ on a set X satisfies:

0-Self $0 = d(x, x)$

Sep $d(x, y) = 0 = d(y, x) \Longrightarrow x = y$

Sym $d(x, y) = d(y, x)$

▽-Inq $d(x, y) + d(y, z) \geq d(x, z)$

Necessarily then:

Pos $d(x, y) \geq 0$

Possible strengthenings:

Bdd $1 \geq d(x, y)$ (*bounded metric*)

Ult $\max\{(d(x, y), d(y, z)\} \geq d(x, z)$ (*ultrametric*)

Met$_{\text{Frechét}}$: morph's $f : X \to Y$ satisfy $d_X(x, x') \geq d_Y(fx, fx')$; write $X(x, x') \geq Y(fx, fx')$.

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1.2 Some shortcomings of $\mathbf{Met}_{\text{Fréchet}}$, Hausdorff's 1914 observations

- Finitely complete, but countable products (even of 2-point spaces) may not exist.
- Any two non-empty spaces fail to admit a coproduct.
- Neither its cartesian structure nor its natural monoidal structure are closed.
- The (non-symmetrized) Hausdorff distance

$$d(A,B) = \sup_{x\in A}\inf_{y\in B} d(x,y)$$

for $A, B \subseteq X$ will (when it exists in $[0,\infty)$) generally satisfy *only* (0-Self) and ($\triangle$-Inq) of the Fréchet axioms, ...

... but this remains true even when the given distance function on X satisfies just these two conditions! Likewise for bounded metrics, ultrametrics, *etc*.

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1.3 Regrouping Fréchet's axioms à la Lawvere 1973

A Frechét metric $d : X \times X \longrightarrow [0, \infty]$ on a set X satisfies

0-*Self-distances*:	$0 \geq d(x,x)$	$1 \to X(x,x)$
∇-*Inequality*:	$d(x,y) + d(y,z) \geq d(y,z)$	$X(x,y) \times X(y,z) \to X(x,z)$
Symmetry:	$d(x,y) = d(y,x)$	$X(x,y) \cong X(y,x)$
Separation:	$d(x,y) = 0 = d(y,x) \Longrightarrow x = y$	$X(x,y) \cong 1 \cong X(y,x) \Longrightarrow x = y$
Finiteness:	$\infty > d(x,y)$	$\emptyset \neq X(x,y)$

A map $f : X \to Y$ of metric spaces is non-expansive / short / 1-Lipschitz if

Contraction:	$d_X(x,x') \geq d_Y(fx,fx')$	$X(x,x') \to Y(fx,fx')$

The category **Met** is complete and cocomplete and symmetric monoidal-closed. But:

$$\begin{tikzcd}
\mathbf{Met}_{\text{Frechét}} \arrow[r, tail, ":-("] & \mathbf{Met}_{\mathrm{sym,sep}} \arrow[r, tail, "\text{reflective}"] & \mathbf{Met}_{\mathrm{sym}} \arrow[rr, tail, "\text{reflective, coreflective}"] \arrow[dr, "\text{topological}"'] & & \mathbf{Met} \arrow[dl, "\text{topological}"] \\
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1.4 Some supporting formulae

- $(f_i : X \to Y_i)_{i\in I}$ initial (jointly cartesian) $\iff X(x,x') = \sup_{i\in I} Y_i(f_i x, f_i x')$
- $X \times Y((x,y),(x',y')) = \max\{X(x,x'), Y(y,y')\}$
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- $X \otimes Y((x,y),(x',y')) = X(x,x') + Y(y,y')$ makes **Met** symmetric monoidal-closed with internal hom
- $[X,Y](f,g) = \sup_{x\in X} Y(fx,gx)$
- Coreflective symmetrization: $X_{\mathrm{csym}}(x,x') = \max\{X(x,x'), X(x',x)\}$
- Reflective symmetrization:

$$X_{\mathrm{rsym}}(x,x') = \inf_{x=x_0,\dots,x_n=x'} \Sigma_{j=1}^{n} \min\{X(x_{j-1},x_j), X(x_j,x_{j-1})\}$$

- Separation: with $(x \simeq y :\iff X(x,y) = 0 = X(y,x))$, let

$$X/\simeq([x],[y]) = X(x,y)$$

... and the formulae remain *essentially* valid for **BMet** (bounded mets), **UMet** (ultramets),

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- Coreflective symmetrization: $X_{\mathrm{csym}}(x,x') = \max\{X(x,x'), X(x',x)\}$
- Reflective symmetrization:

$$X_{\mathrm{rsym}}(x,x') = \inf_{x=x_0,\ldots,x_n=x'} \Sigma_{j=1}^{n} \min\{X(x_{j-1},x_j), X(x_j,x_{j-1})\}$$

- Separation: with $(x \simeq y :\iff X(x,y) = 0 = X(y,x))$, let

$$X/\!\simeq([x],[y]) = X(x,y)$$

... and the formulae remain *essentially* valid for **BMet** (bounded mets), **UMet** (ultramets).

2.1 Quantales

A (commutative) quantale $(\mathcal{V}, \leq, \otimes, \mathrm{k})$ is a commutative monoid in $(\mathbf{Sup}, \boxtimes, 2)$; that is:

- $(\mathcal{V}, \leq)$ is a complete lattice;
- $(\mathcal{V}, \otimes, \mathrm{k})$ is a commutative monoid;
- $- \otimes v : \mathcal{V} \to \mathcal{V}$ preserves joins for all $v \in \mathcal{V}$.

Hence, as a monotone map, every $- \otimes v$ has a right adjoint; this means:

$\mathcal{V}$ is a “thin” symmetric monoidal-closed category, with internal homs $[v, w]$ determined by

$$u \leq [v, w] \iff u \otimes v \leq w.$$

Some useful rules:

$$\mathrm{k} \leq [u, u], \; [u, v] \otimes u \leq v, \; [\mathrm{k}, v] = v, \; [u_1 \otimes u_2, v] = [u_1, [u_2, v]] = [u_2, [u_1, v]],$$

$$[\bigvee_{i \in I} u_i, v] = \bigwedge_{i \in I} [u_i, v], \quad [u, \bigwedge_{i \in I} v_i] = \bigwedge_{i \in I} [u, v_i].$$

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$\varphi : \mathcal{V} \to \mathcal{W}$ is a lax homomorphism if

$$\bigvee_{i\in I} \varphi u_i \le \varphi(\bigvee_{i\in I} u_i), \quad \varphi u \otimes_{\mathcal{W}} \varphi v \le \varphi(u \otimes_{\mathcal{V}} v), \quad \mathrm{k}_{\mathcal{W}} \le \varphi(\mathrm{k}_{\mathcal{V}});$$

φ is a (strict) homomorphism if $\le$ may be replaced by $=$.

- 1 is the terminal quantale ($\mathrm{k} = \bot$)
- $2 = (\{\bot, \top\}, \le, \wedge, \top)$ is the initial quantale; more generally: $(\mathcal{P}S, \subseteq, \cap, S)$ (S any set)
- even more generally: any locale (frame) L is a "cartesian" quantale $(L, \le, \wedge, \top)$
- the Lawvere quantale $[0, \infty]_+ \cong [0, 1]_\times$, that is: $([0, \infty], \ge, +, 0) \cong ([0, 1], \le, \times, 1)$
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2.2 More examples of quantales

- the free quantale $(\mathcal{P}M, \subseteq, *, \{\eta\})$ over a commutative monoid $(M, *, \eta)$
- the quantale $(\mathcal{D}\mathcal{V}, \subseteq, \otimes_{\downarrow}, {\downarrow}\mathrm{k})$ of down(-closed) sets of a quantale $(\mathcal{V}, \otimes, \mathrm{k})$
- the quantale $\Delta_{\&} = (\Delta, \leq, \&, \kappa)$ of distance distribution functions, with

$$\Delta = \{\varphi \colon [0,\infty] \to [0,1] \mid \forall \alpha \in [0,\infty] : \varphi(\alpha) = \sup_{\beta<\alpha} \varphi(\beta)\},$$

for any “t-norm” & on $[0,1]$, *i.e.* any operation that makes $([0,1], \leq, \&, 1)$ a quantale, extended to Δ by

$$(\varphi \,\&\, \psi)(\gamma) = \sup_{\alpha+\beta<\gamma} \varphi(\alpha) \,\&\, \psi(\beta)\,;$$

the distance distribution function κ with $\kappa(0) = 0$ and $\kappa(\alpha) = 1$ for $\alpha > 0$ is &-neutral.

$$\begin{tikzcd}
{[0,\infty]_{+}} \arrow[r, "\sigma_{+}"] & \Delta_{\&} & {[0,1]_{\&}} \arrow[l, "\tau_{\&}"']
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is a coproduct in the category of quantales, since for any $\varphi \in \Delta$:

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2.3 Quantale-valued relations of sets

$$X \xrightarrow{r} \mapsto Y \qquad \Longleftrightarrow \qquad X \times Y \xrightarrow{r} \mathcal{V} \qquad \Longleftrightarrow \qquad r = (r(x,y))_{x \in X, y \in Y}$$

$$(s \cdot r)(x,z) = \bigvee_{y \in Y} r(x,y) \otimes s(y,z)$$

Set $\longrightarrow \mathcal{V}$-**Rel**, $\quad (X \xrightarrow{f} Y) \longmapsto (X \xrightarrow{f_\circ} \mapsto Y) \qquad f_\circ(x,y) = \mathrm{k}$ if $fx = y$, $= \bot$ else

$\mathcal{V}$-**Rel** is a 2-category with 2-cells given by the pointwise order of $\mathcal{V}$-relations.

$\mathcal{V}$-**Rel** has the involution $r^\circ(y,x) = r(x,y)$; put $f^\circ = (f_\circ)^\circ$; then $f_\circ \dashv f^\circ$ ("maps are maps")

$\mathcal{V}$-**Rel** is a quantaloid, *i.e.* a **Sup**-enriched category:

$$\Big(\bigvee_{i \in I} s_i\Big) \cdot r = \bigvee_{i \in I} (s_i \cdot r), \qquad s \cdot \Big(\bigvee_{i \in I} r_i\Big) = \bigvee_{i \in I} (s \cdot r_i)$$

$\mathcal{V} \longrightarrow \mathcal{V}$-**Rel**, $\quad v \longmapsto (1 \xrightarrow{v} \mapsto 1)$ is a homomorphism of quantaloids.

Useful rule: $W \xrightarrow{g} X \xrightarrow{r} \mapsto Y \xleftarrow{h} Z \qquad (h^\circ \cdot r \cdot g_\circ)(w,z) = r(gw, hz)$.

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$$X \overset{r}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Y \qquad \iff \qquad X \times Y \xrightarrow{r} \mathcal{V} \qquad \iff \qquad r = (r(x,y))_{x \in X, y \in Y}$$

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$$\mathbf{Set} \longrightarrow \mathcal{V}\text{-}\mathbf{Rel}, \qquad (X \xrightarrow{f} Y) \longmapsto (X \overset{f_\circ}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Y) \qquad f_\circ(x,y) = \mathrm{k} \text{ if } fx = y, \quad = \bot \text{ else}$$

$\mathcal{V}$-**Rel** is a 2-category with 2-cells given by the pointwise order of $\mathcal{V}$-relations.

$\mathcal{V}$-**Rel** has the involution $r^\circ(y,x) = r(x,y)$; put $f^\circ = (f_\circ)^\circ$; then $f_\circ \dashv f^\circ$ (“maps are maps”)

$\mathcal{V}$-**Rel** is a quantaloid, *i.e.* a **Sup**-enriched category:

$$\Big(\bigvee_{i \in I} s_i\Big) \cdot r = \bigvee_{i \in I} (s_i \cdot r), \qquad s \cdot \Big(\bigvee_{i \in I} r_i\Big) = \bigvee_{i \in I} (s \cdot r_i)$$

$\mathcal{V} \longrightarrow \mathcal{V}$-**Rel**, $\quad v \longmapsto \; (1 \overset{v}{\relbar\joinrel\mapstochar\joinrel\rightarrow} 1)$ is a homomorphism of quantaloids.

Useful rule: $W \xrightarrow{g} X \overset{r}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Y \xleftarrow{h} Z \qquad (h^\circ \cdot r \cdot g_\circ)(w,z) = r(gw, hz).$

2.3 Quantale-valued relations of sets

$$X \overset{r}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Y \qquad \Longleftrightarrow \qquad X \times Y \xrightarrow{r} \mathcal{V} \qquad \Longleftrightarrow \qquad r = (r(x,y))_{x \in X, y \in Y}$$

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$\mathcal{V}$-**Rel** is a 2-category with 2-cells given by the pointwise order of $\mathcal{V}$-relations.

$\mathcal{V}$-**Rel** has the involution $r^\circ(y,x) = r(x,y)$; put $f^\circ = (f_\circ)^\circ$; then $f_\circ \dashv f^\circ$ ("maps are maps")

$\mathcal{V}$-**Rel** is a quantaloid, *i.e.* a **Sup**-enriched category:

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Useful rule: $W \xrightarrow{g} X \overset{r}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Y \xleftarrow{h} Z \qquad (h^\circ \cdot r \cdot g_\circ)(w,z) = r(gw, hz).$

2.3 Quantale-valued relations of sets

$$X \overset{r}{-\!\!\!+\!\!\!\rightarrow} Y \qquad \Longleftrightarrow \qquad X \times Y \xrightarrow{r} \mathcal{V} \qquad \Longleftrightarrow \qquad r = (r(x,y))_{x \in X, y \in Y}$$

$$(s \cdot r)(x,z) = \bigvee_{y \in Y} r(x,y) \otimes s(y,z)$$

$$\textbf{Set} \longrightarrow \mathcal{V}\textbf{-Rel}, \qquad (X \xrightarrow{f} Y) \longmapsto (X \overset{f_\circ}{-\!\!\!+\!\!\!\rightarrow} Y) \qquad f_\circ(x,y) = \mathrm{k} \text{ if } fx = y, \quad = \bot \text{ else}$$

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$$(\bigvee_{i \in I} s_i) \cdot r = \bigvee_{i \in I}(s_i \cdot r), \qquad s \cdot (\bigvee_{i \in I} r_i) = \bigvee_{i \in I}(s \cdot r_i)$$

$\mathcal{V} \longrightarrow \mathcal{V}$-**Rel**, $\quad v \longmapsto (1 \overset{v}{-\!\!\!+\!\!\!\rightarrow} 1)$ is a homomorphism of quantaloids.

Useful rule: $W \xrightarrow{g} X \overset{r}{-\!\!\!+\!\!\!\rightarrow} Y \xleftarrow{h} Z \qquad (h^\circ \cdot r \cdot g_\circ)(w,z) = r(gw, hz).$

2.3 Quantale-valued relations of sets

$$X \overset{r}{\relbar\!\!\!\shortmid\!\!\longrightarrow} Y \qquad \Longleftrightarrow \qquad X \times Y \xrightarrow{r} \mathcal{V} \qquad \Longleftrightarrow \qquad r = (r(x,y))_{x\in X, y\in Y}$$

$$(s \cdot r)(x,z) = \bigvee_{y\in Y} r(x,y) \otimes s(y,z)$$

$$\mathbf{Set} \longrightarrow \mathcal{V}\text{-}\mathbf{Rel}, \qquad (X \xrightarrow{f} Y) \longmapsto (X \overset{f_\circ}{\relbar\!\!\!\shortmid\!\!\longrightarrow} Y) \qquad f_\circ(x,y) = \mathrm{k} \text{ if } fx = y, \quad = \bot \text{ else}$$

$\mathcal{V}$-**Rel** is a 2-category with 2-cells given by the pointwise order of $\mathcal{V}$-relations.

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$\mathcal{V} \longrightarrow \mathcal{V}$-**Rel**, $\quad v \longmapsto (1 \overset{v}{\relbar\!\!\!\shortmid\!\!\longrightarrow} 1)$ is a homomorphism of quantaloids.

Useful rule: $W \xrightarrow{g} X \overset{r}{\relbar\!\!\!\shortmid\!\!\longrightarrow} Y \xleftarrow{h} Z \qquad (h^\circ \cdot r \cdot g_\circ)(w,z) = r(gw, hz).$

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$$X \overset{r}{\relbar\!\!\shortmid\!\!\rightarrow} Y \qquad \Longleftrightarrow \qquad X \times Y \xrightarrow{r} \mathcal{V} \qquad \Longleftrightarrow \qquad r = (r(x,y))_{x\in X, y \in Y}$$

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$\mathcal{V}$-**Rel** is a quantaloid, *i.e.* a **Sup**-enriched category:

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$\mathcal{V} \longrightarrow \mathcal{V}$-**Rel**, $\quad v \longmapsto (1 \overset{v}{\relbar\!\!\shortmid\!\!\rightarrow} 1)$ is a homomorphism of quantaloids.

Useful rule: $W \xrightarrow{g} X \overset{r}{\relbar\!\!\shortmid\!\!\rightarrow} Y \xleftarrow{h} Z \qquad (h^\circ \cdot r \cdot g_\circ)(w,z) = r(gw, hz).$

2.4 Extensions and liftings of $\mathcal{V}$-relations

Consider $X \overset{r}{\relbar\mapstochar\rightarrow} Y$, $Y \overset{s}{\relbar\mapstochar\rightarrow} Z$, $X \overset{t}{\relbar\mapstochar\rightarrow} Z$. Obtain:

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Rel}(Y,Z) \arrow[r, bend left, "- \cdot r"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Rel}(X,Z) \arrow[l, bend left, "{[r,-]}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Rel}(X,Y) \arrow[r, bend left, "s \cdot -"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Rel}(X,Z) \arrow[l, bend left, "{]s,-[}"]
\end{tikzcd}$$

$$s \leq [r,t] \iff s \cdot r \leq t \iff r \leq \,]s,t[$$

"extension of t along r"

$$\begin{tikzcd}
& Z \\
X \arrow[ur, "t", "\shortmid"{marking}] \arrow[r, "r"', "\shortmid"{marking}] & Y \arrow[u, "{[r,t]}"', "\shortmid"{marking}] \arrow[from=2-2, to=1-1, phantom, "\geq" near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "s", "\shortmid"{marking}] & Z \\
X \arrow[u, "{]s,t[}", "\shortmid"{marking}] \arrow[ur, "t"', "\shortmid"{marking}] \arrow[from=1-1, to=2-2, phantom, "\leq" near start] &
\end{tikzcd}$$

"lifting of t along s"

$$[r,t](y,z) = \bigwedge_{x \in X} [r(x,y), t(x,z)] \qquad]s,t[(x,y) = \bigwedge_{z \in Z} [s(y,z), t(x,z)]$$

2.4 Extensions and liftings of $\mathcal{V}$-relations

Consider $X \xrightarrow{r} Y$, $Y \xrightarrow{s} Z$, $X \xrightarrow{t} Z$. Obtain:

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Rel}(Y,Z) \arrow[rr, bend left, "- \cdot r"] \arrow[rr, phantom, "\perp"] & & \mathcal{V}\text{-}\mathbf{Rel}(X,Z) \arrow[ll, bend left, "{[r,-]}"]
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\qquad
\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Rel}(X,Y) \arrow[rr, bend left, "s \cdot -"] \arrow[rr, phantom, "\perp"] & & \mathcal{V}\text{-}\mathbf{Rel}(X,Z) \arrow[ll, bend left, "{]s,-[}"]
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$$s \leq [r,t] \iff s \cdot r \leq t \iff r \leq \,]s,t[$$

"extension of t along r"

$$\begin{tikzcd}
 & Z \\
X \arrow[ur, "t", "\shortmid" marking] \arrow[r, "r"', "\shortmid" marking] \arrow[ur, phantom, "\geq", shift right=3ex] & Y \arrow[u, "{[r,t]}"', "\shortmid" marking]
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "s", "\shortmid" marking] & Z \\
X \arrow[u, "{]s,t[}", "\shortmid" marking] \arrow[ur, "t"', "\shortmid" marking] \arrow[ur, phantom, "\leq", shift left=3ex] &
\end{tikzcd}$$

"lifting of t along s"

$$[r,t](y,z) = \bigwedge_{x \in X} [r(x,y), t(x,z)] \qquad]s,t[(x,y) = \bigwedge_{z \in Z} [s(y,z), t(x,z)]$$

2.5 Small categories and functors enriched in $\mathcal{V}$

$$\begin{array}{lll}
(X,a) \in \mathcal{V}\text{-}\mathbf{Cat} & \iff & a \text{ is a monoid in the monoidal category } (\mathcal{V}\text{-}\mathbf{Rel}(X,X), \leq, \cdot, 1_X^\circ) \\
 & \iff & 1_X^\circ \leq a, \qquad a \cdot a \leq a \\
 & \iff & \mathrm{k} \leq a(x,x), \quad a(x,y) \otimes a(y,z) \leq a(x,z) \\
X \in \mathcal{V}\text{-}\mathbf{Cat} & \iff & \mathrm{k} \leq X(x,x), \quad X(x,y) \otimes X(y,z) \leq X(x,z)
\end{array}$$

$$\begin{array}{lll}
f : X \to Y \text{ in } \mathcal{V}\text{-}\mathbf{Cat} & \iff & X(x,x') \leq Y(fx, fx') \\
f : (X,a) \to (Y,b) & \iff & a \leq f^\circ \cdot b \cdot f_\circ \iff f_\circ \cdot a \leq b \cdot f_\circ \iff a \cdot f^\circ \leq f^\circ \cdot b
\end{array}$$

Some prominent objects in $\mathcal{V}$-**Cat**:

$$\emptyset, \quad 1 = (\{*\}, \top), \quad E = (\{*\}, \mathrm{k}), \quad \mathcal{V} = (\mathcal{V}, [\text{-}, \text{-}])$$

Lax homomorphisms of quantales facilitate change-of-base functors:

$$\begin{array}{l}
\varphi : \mathcal{V} \to \mathcal{W} \text{ lax homomorphism} \qquad \Longrightarrow \mathrm{B}_\varphi : \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{W}\text{-}\mathbf{Cat}, \quad (X,a) \mapsto (X, \varphi a) \\
\mathrm{p} : \mathcal{V} \to 2 \text{ with } (p(v) = \top \iff \mathrm{k} \leq v) \Longrightarrow \mathrm{B}_\mathrm{p} : \mathcal{V}\text{-}\mathbf{Cat} \to \mathbf{Ord} \text{ with } (x \leq y \iff \mathrm{k} \leq X(x,y))
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f:X\to Y \text{ in } \mathcal{V}\text{-}\mathbf{Cat} & \iff & X(x,x')\le Y(fx,fx') \\
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2.6 Some examples of (categories of) $\mathcal{V}$-categories

$\mathsf{1}$-**Cat** $=$ **Set**

$\mathsf{2}$-**Cat** $=$ **Ord**: preordered sets and monotone maps

$[0,\infty]_+$-**Cat** $=$ **Met** $\cong [0,1]_\times$-**Cat** $=$ **ProbOrd**: probabilistic (pre)ordered sets

$[0,\infty]_{\max}$-**Cat** $=$ **UMet** $\cong [0,1]_{\min}$-**Cat**: (Lawvere) ultrametric spaces

$[0,1]_\oplus$-**Cat** $=$ **BMet** $\cong [0,1]_\odot$-**Cat**: bounded (Lawvere) metric spaces

$\Delta_\times$-**Cat** $=$ **ProbMet** probabilistic (Lawvere) metric spaces $(X, p : X \times X \to \Delta)$,
with $p(x,y)(\alpha)$ to be interpreted as probability of "$d(x,y) < \alpha$ *for a random metric on* X"

$$(\mathsf{2} \rightarrowtail [0,\infty]_+ \rightarrowtail \Delta_\times) \qquad \Longrightarrow \qquad (\mathbf{Ord} \rightarrowtail \mathbf{Met} \rightarrowtail \mathbf{ProbMet})$$

$\mathcal{P}(M,*,\eta)$-**Cat** $\ni (X, (\leq_\alpha)_{\alpha\in M})$ with $x \leq_\eta x$, $(x \leq_\alpha y \;\&\; y \leq_\beta z \Longrightarrow x \leq_{\alpha*\beta} z)$

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2.6 Some examples of (categories of) $\mathcal{V}$-categories

1-**Cat** $=$ **Set**

2-**Cat** $=$ **Ord**: preordered sets and monotone maps

$[0,\infty]_{+}$-**Cat** $=$ **Met** $\cong$ $[0,1]_{\times}$-**Cat** $=$ **ProbOrd**: probabilistic (pre)ordered sets

$[0,\infty]_{\mathrm{max}}$-**Cat** $=$ **UMet** $\cong$ $[0,1]_{\mathrm{min}}$-**Cat**: (Lawvere) ultrametric spaces

$[0,1]_{\oplus}$-**Cat** $=$ **BMet** $\cong$ $[0,1]_{\odot}$-**Cat**: bounded (Lawvere) metric spaces

$\Delta_{\times}$-**Cat** $=$ **ProbMet** probabilistic (Lawvere) metric spaces $(X, p : X \times X \to \Delta)$,
with $p(x,y)(\alpha)$ to be interpreted as probability of "$d(x,y) < \alpha$ *for a random metric on* X"

$$(2 \rightarrowtail [0,\infty]_{+} \rightarrowtail \Delta_{\times}) \qquad \Longrightarrow \qquad (\mathbf{Ord} \rightarrowtail \mathbf{Met} \rightarrowtail \mathbf{ProbMet})$$

$$\mathcal{P}(M,*,\eta)\text{-}\mathbf{Cat} \ni (X, (\leq_{\alpha})_{\alpha\in M}) \quad \text{with} \quad x \leq_{\eta} x, \quad (x \leq_{\alpha} y \;\&\; y \leq_{\beta} z \Longrightarrow x \leq_{\alpha*\beta} z)$$

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$$\begin{tikzcd}
(2 \arrow[r, tail] & {[0,\infty]_+} \arrow[r, tail] & \Delta_\times) \arrow[r, phantom, "\Longrightarrow"] & (\mathbf{Ord} \arrow[r, tail] & \mathbf{Met} \arrow[r, tail] & \mathbf{ProbMet})
\end{tikzcd}$$

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2.7 $\mathcal{V}$-**Cat** as a concrete category over **Set**

$\mathcal{V}\text{-}\mathbf{Cat}_X = (\mathrm{O}\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathbf{Set})^{-1}X$ is a complete lattice, with $\bigwedge$ as in $\mathcal{V}\text{-}\mathbf{Rel}(X,X)$, $\bot = 1_X^\circ$

Every $r \in \mathcal{V}\text{-}\mathbf{Rel}(X,X)$ has a $\mathcal{V}\text{-}\mathbf{Cat}_X$-hull $\overline{r} \geq r$: $\overline{r} = \bigvee_{n \geq 0} r^n$.

$\mathrm{O}\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathbf{Set}$ is a bifibration with complete fibres and, hence, a topological functor.

$(f_i : (X,a) \to (Y_i, b_i))_{i \in I}$ initial (= jointly cartesian) $\iff a = \bigwedge_{i \in I} f_i^\circ \cdot b_i \cdot (f_i)_\circ$

$(f_i : (X_i,a_i) \to (Y, b))_{i \in I}$ final (= jointly cocartes'n) $\iff b = \overline{\bigvee_{i \in I} (f_i)_\circ \cdot a_i \cdot f_i^\circ}$

Consequently:

$\mathcal{V}$-**Cat** is complete and cocomplete and O has both adjoints.

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cat} \arrow[r, "\mathrm{O}"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathbf{Set} \arrow[l, bend right=35, "\text{discrete}"'] \arrow[l, bend left=35, "\text{indiscrete}"]
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$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cat} \arrow[r, "\mathrm{O}"] & \mathbf{Set} \arrow[l, bend right=40, "\text{discrete}"', "\perp"] \arrow[l, bend left=40, "\text{indiscrete}", "\perp"']
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$$\begin{tikzcd}
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2.8 $\mathcal{V}$-**Cat** as a closed category and a 2-category

For $X, Y \in \mathcal{V}$-**Cat**, consider $(\mathrm{ev}_x : \mathcal{V}\text{-}\mathbf{Cat}(X,Y) \longrightarrow Y)_{x\in X}$ and put the initial structure on $[X,Y] := \mathcal{V}\text{-}\mathbf{Cat}(X,Y)$:

$$\begin{aligned}[X,Y](f,g) &= \textstyle\bigwedge_{x\in X} Y(fx,gx)\\ &= \textstyle\bigwedge_{x,x'\in X}[X(x,x'),Y(fx,gx')]\end{aligned}$$

The induced (pre)order on $[X,Y]$ is

$$f \le g \iff \forall x\in X : \mathrm{k} \le Y(fx,gx) \iff \forall x \in X : fx \le gx.$$

With its 2-cells given by $\le$, $\mathcal{V}$-**Cat** is thus a 2-category.

Adjunction in $\mathcal{V}$-**Cat**:

$$(X \xrightarrow{f} Y) \dashv (Y \xleftarrow{g} X) \iff X(x,gy) = Y(fx,y) \iff g^{\circ}\cdot a = b\cdot f_{\circ}$$

Note: RHS *forces* f,g to be $\mathcal{V}$-functors and gives $f \dashv g$ in **Ord**, *i.e.* $fg \le 1_Y$ and $1_X \le gf$, but $f \dashv g$ in **Ord** secures $f \dashv g$ in $\mathcal{V}$-**Cat** only when f,g are actually $\mathcal{V}$-functors.

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2.9 $\mathcal{V}$-**Cat** as a symmetric monoidal-closed category, Yoneda

$$X \otimes Y((x,y),(x',y')) = X(x,x') \otimes Y(y,y'), \quad E(*,*) = \mathrm{k}$$

Enriched Universal Property: $\quad [Z \otimes X, Y] \cong [Z,[X,Y]]$

$$\begin{tikzcd}
{[X,Y] \otimes X} \arrow[rr, "\varepsilon"] & & Y \\
Z \otimes X \arrow[u, dashed, "f^\sharp \otimes 1_X"] \arrow[urr, "f"'] & &
\end{tikzcd}
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\begin{tikzcd}
{[X,\mathcal{V}] \otimes X} \arrow[rr, "\varepsilon"] & & \mathcal{V} \\
X^{\mathrm{op}} \otimes X \arrow[u, dashed, "\mathbf{y}_{X^{\mathrm{op}}} \otimes 1_X"] \arrow[urr, "X(-,-)"'] & &
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Yoneda $\mathcal{V}$-functor:

$$\mathbf{y}_X : X \longrightarrow \mathcal{P}_{\mathcal{V}}X = [X^{\mathrm{op}},\mathcal{V}], y \longmapsto X(-,y), \quad \mathbf{y}^\sharp_X : X \longrightarrow \mathcal{P}^\sharp_{\mathcal{V}}X = [X,\mathcal{V}]^{\mathrm{op}}, x \longmapsto X(x,-)$$

Yoneda Lemma:

$$\mathcal{P}_{\mathcal{V}}X(\mathbf{y}_X y, \sigma) = \sigma y, \qquad \mathcal{P}^\sharp_{\mathcal{V}}X(\tau, \mathbf{y}^\sharp_X x) = \tau x$$

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3.1 Distributors (profunctors, (bi)modules))

Slogan: function/relation = functor/distributor

For $\mathcal{V} = [0,\infty]_+$, think of them as "compatible one-way metrics" between two spaces.

Generally:

$$
\begin{aligned}
(X,a) \overset{\rho}{\relbar\!\!\circ\!\!\rightarrow} (Y,b) \quad &\iff b \cdot \rho \cdot a \leq \rho \\
&\iff X(x',x) \otimes \rho(x,y) \otimes Y(y,y') \leq \rho(x',y') \\
&\iff X^{\mathrm{op}}(x,x') \otimes Y(y,y') \leq [\rho(x,y), \rho(x',y')] \\
&\iff \rho : X^{\mathrm{op}} \otimes Y \to \mathcal{V} \quad \text{is a } \mathcal{V}\text{-functor}
\end{aligned}
$$

$\mathcal{V}$-distributors are closed under $\mathcal{V}$-relational composition and under $\bigwedge, \bigvee$ formed in $\mathcal{V}$-**Rel**.

$\mathcal{V}$-**Dist**: objects are $\mathcal{V}$-categories $X = (X,a)$; identity distributor on X: $1_X^* = \ (X \overset{a}{\relbar\!\!\circ\!\!\rightarrow} X)$

$\mathcal{V}$-**Dist** is **Sup**-enriched (a quantaloid) AND also ($\mathcal{V}$-**Cat**)-enriched:

$$\mathcal{V}\text{-}\mathbf{Dist}(X,Y) = [X^{\mathrm{op}} \otimes Y, \mathcal{V}], \qquad \mathcal{V}\text{-}\mathbf{Dist}(X,Y) \otimes \mathcal{V}\text{-}\mathbf{Dist}(Y,Z) \longrightarrow \mathcal{V}\text{-}\mathbf{Dist}(X,Z)$$

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$$\begin{aligned}(X,a) \overset{\rho}{\longrightarrow\!\!\!\!\circ\,} (Y,b) \quad &\iff b\cdot\rho\cdot a \le \rho\\ &\iff X(x',x)\otimes\rho(x,y)\otimes Y(y,y') \le \rho(x',y')\\ &\iff X^{\mathrm{op}}(x,x')\otimes Y(y,y') \le [\rho(x,y),\rho(x',y')]\\ &\iff \rho : X^{\mathrm{op}}\otimes Y \to \mathcal{V} \quad \text{is a } \mathcal{V}\text{-functor}\end{aligned}$$

$\mathcal{V}$-distributors are closed under $\mathcal{V}$-relational composition and under $\bigwedge, \bigvee$ formed in $\mathcal{V}$-**Rel**.

$\mathcal{V}$-**Dist**: objects are $\mathcal{V}$-categories $X = (X,a)$; identity distributor on X: $1_X^* = (X \overset{a}{\longrightarrow\!\!\!\!\circ\,} X)$

$\mathcal{V}$-**Dist** is **Sup**-enriched (a quantaloid) AND also ($\mathcal{V}$-**Cat**)-enriched:

$$\mathcal{V}\text{-}\mathbf{Dist}(X,Y) = [X^{\mathrm{op}}\otimes Y, \mathcal{V}], \qquad \mathcal{V}\text{-}\mathbf{Dist}(X,Y)\otimes\mathcal{V}\text{-}\mathbf{Dist}(Y,Z) \longrightarrow \mathcal{V}\text{-}\mathbf{Dist}(X,Z)$$

3.1 Distributors (profunctors, (bi)modules))

Slogan: function/relation = functor/distributor

For $\mathcal{V} = [0,\infty]_+$, think of them as “compatible one-way metrics” between two spaces.

Generally:

$$
\begin{aligned}
(X,a) \xrightarrow{\rho}\!\!\!\!\!\!\circ\;\; (Y,b) \quad &\iff b \cdot \rho \cdot a \leq \rho \\
&\iff X(x',x) \otimes \rho(x,y) \otimes Y(y,y') \leq \rho(x',y') \\
&\iff X^{\mathrm{op}}(x,x') \otimes Y(y,y') \leq [\rho(x,y), \rho(x',y')] \\
&\iff \rho : X^{\mathrm{op}} \otimes Y \to \mathcal{V} \quad \text{is a } \mathcal{V}\text{-functor}
\end{aligned}
$$

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Generally:

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3.2 $\mathcal{V}$-functors *vs.* $\mathcal{V}$-distributors; extensions, liftings, tensor products

$$(X,a) \xrightarrow{f} (Y,b) \quad \Longrightarrow \quad X \xrightarrow[\circ]{f_* = b \cdot f_\circ} Y\,, \quad Y \xrightarrow[\circ]{f^* = f^\circ \cdot b} X\,, \qquad f_* \dashv f^* \text{ in } \mathcal{V}\textbf{-Dist}$$

$$(-)_* : \mathcal{V}\textbf{-Cat} \longrightarrow \mathcal{V}\textbf{-Dist}^{\mathrm{co}} \qquad\qquad (-)^* : \mathcal{V}\textbf{-Cat} \longrightarrow \mathcal{V}\textbf{-Dist}^{\mathrm{op}}$$

$$f \dashv g \iff f_* = g^* \iff g^* \dashv f^* \iff g_* \dashv f_*$$

$\mathcal{V}$-distributors are closed under the formation of extensions and liftings in $\mathcal{V}$-**Rel**:

“extension of τ along ρ”

$$\begin{tikzcd}
 & Z \\
X \arrow[ur, "\tau", "\circ" marking] \arrow[r, "\rho"', "\circ" marking] \arrow[ur, phantom, "\geq", shift right=3ex] & Y \arrow[u, "{[\rho,\tau]}"', "\circ" marking]
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "\sigma", "\circ" marking] & Z \\
X \arrow[u, "{]\sigma,\tau[}", "\circ" marking] \arrow[ur, "\tau"', "\circ" marking] \arrow[ur, phantom, "\leq", shift left=3ex] &
\end{tikzcd}$$

“lifting of τ along σ”

$$\mathcal{V}\textbf{-Dist}(Y,Z)(\sigma,[\rho,\tau]) = \mathcal{V}\textbf{-Dist}(X,Z)(\sigma\cdot\rho,\tau) = \mathcal{V}\textbf{-Dist}(X,Y)(\rho,]\sigma,\tau[)$$

$\mathcal{V}$-**Dist** is symmetric monoidal: $\rho\otimes\varphi : X\otimes S \to Y\otimes T$

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$$\begin{tikzcd}
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X \arrow[ru, "\circ" marking, "\tau"] \arrow[r, "\circ" marking, "\rho"'] \arrow[r, phantom, shift left=4, "\geq"] & Y \arrow[u, "\circ" marking, "{[\rho,\tau]}"']
\end{tikzcd}
\qquad
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Y \arrow[r, "\circ" marking, "\sigma"] & Z \\
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$$\begin{tikzcd}(X,a) \arrow[r, "f"] & (Y,b) & \Longrightarrow & X \arrow[r, "f_*=b\cdot f_\circ", "\circ" marking] & Y, & Y \arrow[r, "f^*=f^\circ\cdot b", "\circ" marking] & X, & f_* \dashv f^* \text{ in } \mathcal{V}\textbf{-Dist}\end{tikzcd}$$

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$$(-)_* : \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Dist}^{\mathrm{co}} \qquad\qquad (-)^* : \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Dist}^{\mathrm{op}}$$

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3.3 $\mathcal{V}$-distributors *vs.* $\mathcal{V}$-presheaves

$$\mathcal{V}\text{-}\mathbf{Dist}(X,Y) = [X^{\mathrm{op}} \otimes Y, \mathcal{V}] \cong [Y, \mathcal{P}_{\mathcal{V}}X] \cong [X^{\mathrm{op}}, [Y,\mathcal{V}]] \cong [X, \mathcal{P}^{\sharp}_{\mathcal{V}}Y]^{\mathrm{op}}$$

$$\mathcal{P}_{\mathcal{V}}X \cong \mathcal{V}\text{-}\mathbf{Dist}(X,E) \qquad \mathcal{P}^{\sharp}_{\mathcal{V}}Y \cong (\mathcal{V}\text{-}\mathbf{Dist}(E,Y))^{\mathrm{op}}$$

The Fundamental Presheaf Adjunction:

$$\begin{tikzcd} \mathcal{V}\text{-}\mathbf{Dist}^{\mathrm{op}} \arrow[r, bend right, "\mathcal{P}_{\mathcal{V}} \cong \mathcal{V}\text{-}\mathbf{Dist}(-,E)"'] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Cat} \arrow[l, bend right, "(-)^{*}"'] \end{tikzcd}$$

For $X \overset{\rho}{\longrightarrow\!\!\!\!\!\circ\;\,} Y$:
$$\mathcal{P}_{\mathcal{V}}\rho : \mathcal{P}_{\mathcal{V}}Y \longrightarrow \mathcal{P}_{\mathcal{V}}X, \quad (Y \overset{\sigma}{\longrightarrow\!\!\!\!\!\circ\;\,} E) \longmapsto (X \overset{\sigma\cdot\rho}{\longrightarrow\!\!\!\!\!\circ\;\,} E)$$

$$(\mathcal{P}_{\mathcal{V}}\rho)(\sigma)(x) = \bigvee_{y \in Y} \rho(x,y) \otimes \sigma(y)$$

adjunction units: $\mathbf{y}_X : X \longrightarrow \mathcal{P}_{\mathcal{V}}X$, adjunction counits: $(\mathbf{y}_X)_* : X \longrightarrow\!\!\!\!\!\circ\;\, \mathcal{P}_{\mathcal{V}}X$

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The Fundamental Presheaf Adjunction:

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Dist}^{\mathrm{op}} \arrow[r, bend right=15, "\mathcal{P}_{\mathcal{V}} \cong \mathcal{V}\text{-}\mathbf{Dist}(-{,}E)"'] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Cat} \arrow[l, bend right=15, "(-)^{*}"']
\end{tikzcd}$$

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$$(\mathcal{P}_{\mathcal{V}}\rho)(\sigma)(x) = \bigvee_{y\in Y} \rho(x,y) \otimes \sigma(y)$$

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3.4 The $\mathcal{V}$-presheaf monad $(\mathcal{P}, \mathbf{s}, \mathbf{y})$ and its discretization

$$\mathcal{P} : \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}, \quad (f : X \to Y) \longmapsto (\mathcal{P}_{\mathcal{V}} f^* : \mathcal{P}_{\mathcal{V}} X \to \mathcal{P}_{\mathcal{V}} Y,\ \sigma \mapsto \sigma \cdot f^*)$$

$$(\mathcal{P}f)(\sigma)(y) = \bigvee_{x \in X} Y(y, fx) \otimes \sigma x$$

$$\mathbf{s}_X : \mathcal{P}\mathcal{P}X \longrightarrow \mathcal{P}X \qquad \mathbf{s}_X(\Sigma) = \Sigma \cdot (\mathbf{y}_X)_*, \quad \mathbf{s}_X(\Sigma)(x) = \bigvee_{\sigma \in \mathcal{P}_{\mathcal{V}} X} \Sigma(\sigma) \otimes \sigma(x)$$

$(\mathcal{P}, \mathbf{s}, \mathbf{y})$ is a 2-monad, with $\mathcal{P}$ locally fully faithful: $(f \le g \iff \mathcal{P}f \le \mathcal{P}g)$

$(\mathcal{P}, \mathbf{s}, \mathbf{y})$ is lax idempotent (Kock–Zöberlein): $\mathcal{P}\mathbf{y}_X \le \mathbf{y}_{\mathcal{P}X}$

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\mathcal{V}\text{-}\mathbf{Dist}^{\mathrm{op}} \arrow[r, bend right, "\mathcal{P}_{\mathcal{V}} \cong \mathcal{V}\text{-}\mathbf{Dist}(-{,}E)"'] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Cat} \arrow[l, bend right, "(-)^*"'] \arrow[r, bend right, "\mathrm{o}"'] & \mathbf{Set} \arrow[l, bend right, "\mathrm{d(iscrete)}"']
\end{tikzcd}$$

$$\mathcal{P}_{\mathrm{d}} : \mathbf{Set} \longrightarrow \mathbf{Set} \qquad (f : X \to Y) \longmapsto (\mathcal{P}_{\mathrm{d}} f : \mathcal{V}^X \to \mathcal{V}^Y,\ \sigma \mapsto \sigma \cdot f^\circ)$$

$$(\mathcal{P}_{\mathrm{d}} f)(\sigma)(y) = \bigvee_{x \in f^{-1}y} \sigma(x)$$

$$(\mathbf{y}_{\mathrm{d}})_X : X \longrightarrow \mathcal{P}_{\mathrm{d}} X \qquad (\mathbf{y}_{\mathrm{d}})_X(y) = \mathbf{y}_{X_{\mathrm{d}}}(y) = 1_X^\circ(-, y)$$

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3.4 The $\mathcal{V}$-presheaf monad $(\mathcal{P}, \mathbf{s}, \mathbf{y})$ and its discretization

$\mathcal{P} : \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$, $\quad (f : X \to Y) \longmapsto (\mathcal{P}_\mathcal{V} f^* : \mathcal{P}_\mathcal{V} X \to \mathcal{P}_\mathcal{V} Y,\ \sigma \mapsto \sigma \cdot f^*)$

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3.5 Distributors are Kleisli morphisms

$$\begin{tikzcd} X \arrow[r, "\varphi", "\circ" marking] & Y & \Longleftrightarrow & Y \arrow[r, "\varphi^\sharp"] & \mathcal{P}X \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "1_X^*", "\circ" marking] & X & \Longleftrightarrow & X \arrow[r, "\mathsf{y}_X"] & \mathcal{P}X \end{tikzcd}$$

$$\Big(\begin{tikzcd} X \arrow[r, "\varphi", "\circ" marking] & Y \arrow[r, "\psi", "\circ" marking] & Z \end{tikzcd} \Big)^\sharp = \Big(\begin{tikzcd} Z \arrow[r, "\psi^\sharp"] & \mathcal{P}Y \arrow[r, "\mathcal{P}\varphi^\sharp"] & \mathcal{P}\mathcal{P}X \arrow[r, "\mathsf{s}_X"] & \mathcal{P}X \end{tikzcd} \Big)$$

$$(\mathcal{V}\text{-}\mathbf{Dist})^{\mathrm{op}} \cong \mathrm{Kl}(\mathcal{P})$$

$$(\mathcal{V}\text{-}\mathbf{Rel})^{\mathrm{op}} \cong \mathrm{Kl}(\mathcal{P}_{\mathrm{d}})$$

Q: What “is” $\mathrm{EM}(\mathcal{P})$?

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$$(\,X \overset{\varphi}{\relbar\!\!\circ\!\!\rightarrow} Y \overset{\psi}{\relbar\!\!\circ\!\!\rightarrow} Z\,)^\sharp = (\,Z \xrightarrow{\psi^\sharp} \mathcal{P}Y \xrightarrow{\mathcal{P}\varphi^\sharp} \mathcal{P}\mathcal{P}X \xrightarrow{\mathbf{s}_X} \mathcal{P}X\,)$$

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4.1 Order completeness *vs.* conical (co)completeness

Consider $X \in \mathcal{V}$-**Cat** with its induced order ($x \leq y \iff \mathrm{k} \leq X(x,y)$). Then:

$y \simeq \bigwedge_{i \in I} x_i \iff \forall z\ (\mathrm{k} \leq X(z,y) \iff \forall i \in I : \mathrm{k} \leq X(z,x_i))$

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$\iff \mathbf{y}_X y = \bigwedge_{i \in I} \mathbf{y}_X x_i$ in $\mathcal{P}_{\mathcal{V}} X = [X^{\mathrm{op}}, \mathcal{V}]$

X conically complete $\iff$ X has all conical infima

$\iff$ X has all infima and $\mathbf{y}_X$ preserves them

$\iff$ X is order-complete, $\mathbf{y}_X$ is an inf-map

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4.2 Conical completeness: examples and remarks

In $X \in 2$-**Cat** $=$ **Ord**, every inf/sup is conical; hence:
X conically (co)complete $\iff$ X order-complete

In $X \in [0,\infty]$-**Cat** $=$ **Met**, not even binary infs/sups have to be conical.
The order of $X \in \mathbf{Met}_{\mathrm{sym,sep}}$ is discrete, so that X is order-complete only when $|X| = 1$.

$\mathcal{V} \in \mathcal{V}$-**Cat** is always conically (co)complete, and so is $\mathcal{P}_{\mathcal{V}}X$, for all $X \in \mathcal{V}$-**Cat**.

For $\mathcal{V}$ non-integral ($\mathrm{k} < \top$), one finds $X \in \mathcal{V}$-**Cat** order-compl., but not conically (co)compl.

There is a subspace of $[0,\infty] \in [0,\infty]_+$-**Cat** is conically complete category but fails to be conically cocomplete (Clementino).

We need:
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In $X \in [0,\infty]$-**Cat** $=$ **Met**, not even binary infs/sups have to be conical.
The order of $X \in$ **Met**$_{\mathrm{sym,sep}}$ is discrete, so that X is order-complete only when $|X| = 1$.

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For $\mathcal{V}$ non-integral ($\mathrm{k} < \top$), one finds $X \in \mathcal{V}$-**Cat** order-compl., but not conically (co)compl.

There is a subspace of $[0,\infty] \in [0,\infty]_+$-**Cat** is conically complete category but fails to be conically cocomplete (Clementino).

We need:
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4.3 Tensored and cotensored $\mathcal{V}$-categories

Recall:
X conically complete $\iff$ X order complete and ($\forall x \in X : X(x,-) : X \to \mathcal{V}$ pres. infs)

Definition:
X tensored $\quad :\iff \forall x \in X : X(x,-) : X \to \mathcal{V}$ has a left adjoint $- \odot x : \mathcal{V} \to X$

$$X(u \odot x, y) = [u, X(x,y)] \qquad (*)$$

X cotensored $:\iff \forall y \in X : X(-,y) : X^{\mathrm{op}} \to \mathcal{V}$ has a left adjoint $- \pitchfork y : \mathcal{V} \to X$

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Note:
Necessarily $u \odot x = \bigwedge\{y \in X \mid u \leq X(x,y)\}$,
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4.4: Tensored $\mathcal{V}$-categories: examples and remarks

$\emptyset \neq X \in 2$-**Cat** $=$ **Ord** is tensored $\iff$ X has a least element.

$\mathcal{V} \in \mathcal{V}$-**Cat** is tensored and cotensored, with $u \odot x = u \otimes x$ and $u \pitchfork x = [u, x]$.
More generally, $\mathcal{P}_{\mathcal{V}}X$ is (co-)tensored, for every $\mathcal{V}$-category X.

A full $\mathcal{V}$-subcategory of $\mathcal{V} \in \mathcal{V}$-**Cat** may fail to be tensored or cotensored.

$[0, \infty]_{\mathrm{csym}}$ $(= [0, \infty]$ with the Euclidean metric) fails to be tensored or cotensored in **Met**.

Products of (co)tensored $\mathcal{V}$-categories are (co)tensored.

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4.5.1 Presenting tensored $\mathcal{V}$-categories via the action of $\mathcal{V}$: prelims

Rules for the action of $\mathcal{V}$ on a tensored $\mathcal{V}$-category X:

(1) $\mathrm{k} \odot x \simeq x$

(2) $(u \otimes v) \odot x \simeq u \odot (v \odot x)$

(3) $(\bigvee_{i \in I} u_i) \odot x \simeq \bigvee_{i \in I}(u_i \odot x)$ (with the RHS $\bigvee$ existing in X, as part of the condition)

(4^-) $x \leq y \Longrightarrow u \odot x \leq u \odot y$

Conversely:
Let X be just a preordered set equipped with a map $\odot : \mathcal{V} \times X \longrightarrow X$ satisfying $(1) - (4^-)$.
Then, for every $x \in X$, the map $- \odot x : \mathcal{V} \longrightarrow X$ has a right adjoint $X(x, -)$, defined by

$$X(x, y) = \bigvee \{u \mid u \odot x \leq y\},$$

making X a $\mathcal{V}$-category, whose underlying preorder is the given one and,
by the given rules and adjunction, satisfies

$$X(u \odot x, y) = [u, X(x, y)],$$

making X a tensored $\mathcal{V}$-category.

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4.5.2 Presenting tensored $\mathcal{V}$-categories via the action of $\mathcal{V}$: theorem

Theorem (Martinelli 2021)

There is a 2-equivalence

$$\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{tensor}} \simeq \mathbf{Ord}^{\mathcal{V}}_{\frac{1}{2}\mathrm{cocts}}$$

$\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{tensor}}$*:*
small tensored $\mathcal{V}$-categories, with tensor-preserving $\mathcal{V}$-functors

$\mathbf{Ord}^{\mathcal{V}}_{\frac{1}{2}\mathrm{cocts}}$*:*
preordered sets on which $\mathcal{V}$ acts, satisfying conditions $(1), (2), (3), (4^-)$, with monotone and pseudo-equivariant maps.

4.6 Weighted colimits and limits: definitions

Given a “diagram” $Z \xrightarrow{h} X$ in X and “weights” $Z \overset{\omega}{\longrightarrow\!\!\!\!\!\!\circ\ \ } W$ and $W \overset{\upsilon}{\longrightarrow\!\!\!\!\!\!\circ\ \ } Z$. Then:

$$(W \xrightarrow{q} X) \simeq \mathrm{colim}^{\omega} h :\iff q_* = [\omega, h_*], \qquad (W \xrightarrow{p} X) \simeq \lim{}^{\upsilon} h :\iff p^* =]\upsilon, h^*[$$

$$\begin{tikzcd}
 & X \\
Z \arrow[ur, "h_*", "\circ"{marking}] \arrow[r, "\omega"', "\circ"{marking}] & W \arrow[u, "{q_*=[\omega,h_*]}"', "\circ"{marking}] \arrow[ul, phantom, "\geq"]
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "\upsilon", "\circ"{marking}] & Z \\
X \arrow[u, "{p^*=]\upsilon,h^*[}", "\circ"{marking}] \arrow[ur, "h^*"', "\circ"{marking}] \arrow[ur, phantom, "\leq", shift left=3]
\end{tikzcd}$$

$$X(qt, x) = \bigwedge_{z \in Z} [\omega(z,t), X(hz, x)] \qquad X(x, pt) = \bigwedge_{z \in Z} [\upsilon(t,z), X(x, hz)]$$

for all $x \in X, t \in W$.

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4.7 Tensors and conical sups as weighted colimits, and conversely

Let $(x \in X \iff x : E = (\{*\}, \mathrm{k}) \to X)$ and $(f = (x_i)_{i\in I}$ in $X \iff f : I_{\mathrm{d}} \cong \coprod_{i\in I} E \to X)$, let $\nabla : \coprod_{i\in I} E \to E$ be the "codiagonal". Then:

$$u \odot x \simeq \mathrm{colim}^u x, \quad \bigvee_{i\in I}^{\nabla} x_i \simeq \mathrm{colim}^{\nabla_*} f, \quad u \pitchfork x \simeq \lim{}^u x, \quad \bigwedge_{i\in I}^{\nabla} x_i \simeq \lim{}^{\nabla^*} f.$$

Theorem

Let $Z \xrightarrow{h} X$ be a diagram in the tensored $\mathcal{V}$-category X with weight $Z \overset{\omega}{\relbar\!\!\circ\!\!\rightarrow} W$. Then

$$(\mathrm{colim}^{\omega} h)(t) \simeq \bigvee_{z\in Z}^{\nabla} \omega(z,t) \odot h(z)$$

for all $t \in W$, with the colimit on the left existing precisely when the conical supremum on the right exists in X for all $t \in W$.

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4.8 Hiding the diagram in the weight

Corollary

1. *X cocomplete $\iff$ X is tensored and conically cocomplete;*
2. *X complete $\iff$ X is cotensored and conically complete.*
3. *X complete and cocomplete $\iff$ X tensored, cotensored and order-complete.*

Given a diagram $Z \xrightarrow{h} X$ in X and weights $Z \overset{\omega}{\longrightarrow\!\!\!\!\!\circ\,} W$ and $W \overset{\upsilon}{\longrightarrow\!\!\!\!\!\circ\,} Z$. Then

$$\mathrm{colim}^{\omega} h \simeq \mathrm{colim}^{\omega \cdot h^*} 1_X \qquad \text{and} \qquad \lim^{\upsilon} h \simeq \lim^{h_* \cdot \upsilon} 1_X,$$

with the (co)limit on either side of $\simeq$ existing when the (co)limit on the other side exists.
In particular:

$$u \odot x \cong \mathrm{colim}^{u} x \cong \mathrm{colim}^{u \cdot x^*} 1_X = \mathrm{colim}^{u \cdot \mathbf{y}_X x} 1_X \ \text{ and } \ \bigvee_{i \in I}^{\triangledown} x_i \simeq \mathrm{colim}^{\omega} 1_X, \ \text{with } \omega = \bigvee_{i \in I} \mathbf{y}_X x_i.$$

Hence: It suffices to let $Z = X$, $h = 1_X$ and $W = E$; presheaves on X suffice as weights!

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4.9.1 Preservation of (co)limits: definition

Definition

Let $h : Z \to X,\ f : X \to Y$ be $\mathcal{V}$-functors and $Z \xrightarrow{\omega} W \xrightarrow{\upsilon} Z$ be $\mathcal{V}$-distributors.

1. If $q \simeq \mathrm{colim}^{\omega} h$ exists in X, one says that $f : X \to Y$ preserves the colimit if the colimit $\mathrm{colim}^{\omega}(f \cdot h)$ exists in Y and is given by $f \cdot q$; equivalently, if one has the implication

$$q_* = [\omega, h_*] \quad \Longrightarrow \quad (f \cdot q)_* = [\omega, (f \cdot h)_*].$$

2. Dually, if $p \simeq \lim^{\upsilon} h$ exists in X, one says that $f : X \to Y$ preserves the limit if the limit $\lim^{\upsilon}(f \cdot h)$ exists in Y and is given by $f \cdot p$; equivalently, if one has the implication

$$p^* =]\upsilon, h^*[\quad \Longrightarrow \quad (f \cdot p)^* =]\upsilon, (f \cdot h)^*[.$$

3. The $\mathcal{V}$-functor f is (co)continuous if it preserves all existing (co)limits in X.

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4.9.2 Preservation of (co)limits: criteria, examples

Let $f : X \to Y,\ g : Y \to X,\ h : Z \to [X, Y]$ be $\mathcal{V}$-functors, $x \in X,\ Z \overset{\omega}{\longrightarrow\!\!\!\!\!\circ\!\!\!\!\to} W$.

1. If X is tensored: f is cocontinuous $\iff$ f preserves tensors and conical suprema.
2. If X is cotensored: f is continuous $\iff$ f preserves cotensors and conical infima.
3. $X(x, -) : X \to \mathcal{V}$ is continuous, $X(-, x) : X \to \mathcal{V}^{\mathrm{op}}$ is cocontinuous.
4. $\mathrm{colim}^{\omega}(h : Z \to [X, Y])$ exists if $\mathrm{colim}^{\omega}\mathrm{ev}_x h$ exists in Y for all x, and it is then preserved by every $\mathrm{ev}_x : [X, Y] \to Y$.
5. $\mathbf{y}_X : X \to \mathcal{P}_{\mathcal{V}}X = [X^{\mathrm{op}}, \mathcal{V}]$ is continuous, $\mathbf{y}^{\sharp}_X : X \to \mathcal{P}^{\sharp}_{\mathcal{V}}X = [X, \mathcal{V}]^{\mathrm{op}}$ is cocontinuous.
6. If $f \dashv g$, then g is continuous and f is cocontinuous.

Theorem (Adjoint Functor Theorem)

1. *Y complete: $g : Y \to X$ has a left adjoint $\mathcal{V}$-functor $\iff$ g is continuous.*
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4.10 Completeness Theorem

Theorem

For every $\mathcal{V}$-category X, the following statements are equivalent:

- (i) *X is cocomplete;*
- (ii) *for every presheaf ω on X, the colimit of 1_X weighted by ω exists in X;*
- (iii) *$\mathbf{y}_X : X \to [X^{\mathrm{op}}, \mathcal{V}]$ has a left adjoint $\mathcal{V}$-functor;*
- (iv) *X is tensored, cotensored and order-complete;*
- (v) *X is complete;*
- (vi) *for every copresheaf υ on X, the limit of 1_X weighted by υ exists in X;*
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5.1 $\mathcal{V}$ is injective in $\mathcal{V}$-**Cat**

$f : X \to Y$ fully faithful $\iff$ $f^* \cdot f_* = 1_X^*$ $\iff$ $X(x, x') = Y(fx, fx')$ for all $x, x' \in X$

$$\begin{tikzcd}
 & \mathcal{V} \\
X \arrow[ur, "\varphi"] \arrow[r, "f"'] & Y \arrow[u, "\psi"']
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
Y \arrow[r, "f^*", "\circ"{marking}] & X \\
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Given f fully faithful and φ, there is a least and a largest extension, ψ_- and ψ^-:

$\psi_- = f_* \cdot \varphi$ and $\psi^- = [(f^*)^\circ, \varphi] = [(f^{\mathrm{op}})_*, \varphi]$

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5.2 $\mathcal{V}$ is an injective regular cogenerator in $\mathcal{V}$-**Cat**$_{\mathrm{sep}}$

$$f \neq g : X \to Y,\ \ Y \text{ separated } \Longrightarrow \exists h : Y \to \mathcal{V} : hf \neq hg$$

$$\kappa_Y : Y \longrightarrow \mathcal{V}^{[Y,\mathcal{V}]} = \prod_{h \in [Y,\mathcal{V}]} \mathcal{V}, \quad y \longmapsto (hy)_{h \in [Y,\mathcal{V}]}$$

$$\pi_Y : \mathcal{V}^{[Y,\mathcal{V}]} \longrightarrow \mathcal{V}^Y, \quad (v_h)_{h \in [Y,\mathcal{V}]} \longmapsto (v_{\mathbf{y}_Y^\sharp z})_{z \in Y}$$

Theorem

*$\mathcal{V}$ is a regular cogenerator of the category $\mathcal{V}$-**Cat**$_{\mathrm{sep}}$, and it is injective with respect to fully faithful $\mathcal{V}$-functors. Every separated $\mathcal{V}$-category Y embeds fully into the Y-fold power $\mathcal{V}^Y$ of $\mathcal{V}$, which is injective again.*

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5.3.1 Colimit and limit completion of a $\mathcal{V}$-category

Every $\mathcal{V}$-presheaf ω on $X \in \mathcal{V}$-**Cat** is a colimit of $\mathbf{y}_X$ in $\mathcal{P}_{\mathcal{V}}X$ weighted by ω: $\omega \simeq \mathrm{colim}^{\omega}\mathbf{y}_X$.

$$\begin{tikzcd}
 & \mathcal{P}_{\mathcal{V}}X \\
X \arrow[ur, "\circ" marking, "(\mathbf{y}_X)_*"] \arrow[r, "\circ" marking, "\omega"'] \arrow[ur, phantom, shift right=4, "\geq"] & E \arrow[u, "\circ" marking, "\omega_*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\circ" marking, "\upsilon"] & X \\
\mathcal{P}^{\sharp}_{\mathcal{V}}X \arrow[u, "\circ" marking, "\upsilon^*"] \arrow[ur, "\circ" marking, "(\mathbf{y}^{\sharp}_X)^*"'] \arrow[ur, phantom, shift left=4, "\leq"] &
\end{tikzcd}$$

Dually, every $\mathcal{V}$-copresheaf υ on X is a limit of representables in $\mathcal{P}^{\sharp}_{\mathcal{V}}X$; that is: $\upsilon \simeq \lim^{\upsilon}\mathbf{y}^{\sharp}_X$.

Wanted for $f : X \to Y$, Y cocomplete/complete:

$$\begin{tikzcd}
X \arrow[r, "\mathbf{y}_X"] \arrow[dr, "f"'] & \mathcal{P}_{\mathcal{V}}X \arrow[d, dashed, "\tilde{f}\ \text{cocontinuous}"] \arrow[dl, phantom, "\simeq"] \\
 & Y
\end{tikzcd}
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\begin{tikzcd}
X \arrow[r, "\mathbf{y}^{\sharp}_X"] \arrow[dr, "f"'] & \mathcal{P}^{\sharp}_{\mathcal{V}}X \arrow[d, dashed, "\bar{f}\ \text{continuous}"] \arrow[dl, phantom, "\simeq"] \\
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\end{tikzcd}$$

5.3.2 Proof of the Colimit Completion Theorem

Uniqueness:

$$\tilde{f}(\omega) \simeq \tilde{f}(\mathrm{colim}^{\omega}\mathbf{y}_X) \simeq \mathrm{colim}^{\omega}(\tilde{f}\mathbf{y}_X) \simeq \mathrm{colim}^{\omega} f$$

Existence:

$$\tilde{f}(\omega) = \mathrm{colim}^{\omega} f \simeq \mathrm{colim}^{\omega \cdot f^*} 1_Y$$

$$\tilde{f} \simeq (\mathrm{colim}^{(\text{-})} 1_Y)(\mathcal{P}_{\mathcal{V}} f^*)$$

$$\tilde{f}\mathbf{y}_X \simeq (\mathrm{colim}^{(\text{-})} 1_Y)(\mathcal{P}_{\mathcal{V}} f^*)\mathbf{y}_X \simeq (\mathrm{colim}^{(\text{-})} 1_Y)\mathbf{y}_Y f \simeq f.$$

$$\begin{tikzcd}
X \arrow[r, "\mathbf{y}_X"] \arrow[d, "f"'] & \mathcal{P}_{\mathcal{V}} X \arrow[dl, "\tilde{f}"'] \arrow[d, "\mathcal{P}_{\mathcal{V}} f^*"] \\
Y \arrow[r, bend right, "\mathbf{y}_Y"'] \arrow[r, phantom, "\perp"] & \mathcal{P}_{\mathcal{V}} Y \arrow[l, bend right, "\mathrm{colim}^{(\text{-})} 1_Y"']
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$\tilde{f}$ is cocontinuous, as the composite of two left adjoints!

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Y \arrow[r, bend right, "\mathbf{y}_Y"'] \arrow[r, phantom, "\perp"] & \mathcal{P}_{\mathcal{V}} Y \arrow[l, bend right, "\mathrm{colim}^{(\text{-})} 1_Y"']
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$\tilde{f}$ is cocontinuous, as the composite of two left adjoints!

5.3.2 Proof of the Colimit Completion Theorem

Uniqueness:

$$\tilde{f}(\omega) \simeq \tilde{f}(\mathrm{colim}^{\omega}\mathbf{y}_X) \simeq \mathrm{colim}^{\omega}(\tilde{f}\mathbf{y}_X) \simeq \mathrm{colim}^{\omega} f$$

Existence:

$$\tilde{f}(\omega) = \mathrm{colim}^{\omega} f \simeq \mathrm{colim}^{\omega \cdot f^*} 1_Y$$

$$\tilde{f} \simeq (\mathrm{colim}^{(\text{-})} 1_Y)(\mathcal{P}_{\mathcal{V}} f^*)$$

$$\tilde{f}\mathbf{y}_X \simeq (\mathrm{colim}^{(\text{-})} 1_Y)(\mathcal{P}_{\mathcal{V}} f^*)\mathbf{y}_X \simeq (\mathrm{colim}^{(\text{-})} 1_Y)\mathbf{y}_Y f \simeq f.$$

$$\begin{tikzcd}
X \arrow[r, "\mathbf{y}_X"] \arrow[d, "f"'] & \mathcal{P}_{\mathcal{V}} X \arrow[d, "\mathcal{P}_{\mathcal{V}} f^*"] \arrow[dl, "\tilde{f}"'] \\
Y \arrow[r, bend right, "\mathbf{y}_Y"'] \arrow[r, phantom, "\perp"] & \mathcal{P}_{\mathcal{V}} Y \arrow[l, bend right, "\mathrm{colim}^{(\text{-})} 1_Y"']
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5.4 Cocomplete $\mathcal{V}$-categories as pseudo-algebras and as injectives

Theorem

The following properties for a $\mathcal{V}$-category X are equivalent:

(i) *X is (co)complete;*

(ii) *X carries the structure of a pseudo-algebra with respect to the presheaf monad on $\mathcal{V}$-**Cat**;*

(iii) *The Yoneda $\mathcal{V}$-functor $\mathbf{y}_X$ has a pseudo-retraction; that is: there is a $\mathcal{V}$-functor $h : \mathcal{P}_{\mathcal{V}}X \to X$ with $h\,\mathbf{y}_X \simeq 1_X$;*

(iv) *X is pseudo-injective in $\mathcal{V}$-**Cat** with respect to fully faithful functors.*

5.5.1 Cocomplete $\mathcal{V}$-categories via cocontinuous action

Let X be a (co)complete preordered set equipped with a map $\odot : \mathcal{V} \times X \longrightarrow X$ satisfying

(1) $\mathrm{k} \odot x \simeq x$

(2) $(u \otimes v) \odot x \simeq u \odot (v \odot x)$

(3) $(\bigvee_{i \in I} u_i) \odot x \simeq \bigvee_{i \in I}(u_i \odot x)$

(4) $u \odot (\bigvee_{i \in I} x_i) \simeq \bigvee_{i \in I}(u \odot x_i)$

Condition (4) (= sup-preservation of every $u \odot - : X \longrightarrow X$) makes the (existing) sups in X conical colimits:

$$X(\bigvee_{i \in I} x_i, y) = \bigwedge_{i \in I} X(x_i, y).$$

Combine this with two fundamental enriched colimit formulae we have already seen:

$$(\mathrm{colim}^{\omega} h)(w) \simeq \bigvee_{z} \omega(z, w) \odot h(z) \qquad (h : Z \to X,\ \omega : Z^{\mathrm{op}} \otimes W \to \mathcal{V})$$

$X(\mathrm{colim}^{\omega} 1_X, x) \simeq [X^{\mathrm{op}}, \mathcal{V}](\omega, \mathbf{y}_X x) \qquad (\omega : X^{\mathrm{op}} \cong X^{\mathrm{op}} \otimes E \to \mathcal{V})$, saying $\mathrm{colim}^{(\text{-})} \dashv \mathbf{y}_X$,

to obtain:

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5.5.2 Cocomplete $\mathcal{V}$-categories via cocontinuous action: Theorem

Theorem (Folklore 19??)

There are 2-equivalences

$$\mathcal{V}\text{-}\mathbf{Cat}^{\mathcal{P}_{\simeq}} \simeq \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{colim}} \simeq (\mathbf{Ord}_{\mathrm{sup}})^{\mathcal{V}}$$

$\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{colim}}$*:*
(co)complete $\mathcal{V}$-categories, with cocontinuous $\mathcal{V}$-functors

$\mathbf{Ord}^{\mathcal{V}}_{\mathrm{sup}}$*:*
(co)complete *preordered sets on which $\mathcal{V}$ acts satisfying conditions* (1), (2), (3), (4)*, with sup-preserving and pseudo-equivariant maps*

Corollary

There are 2-equivalences

$$(\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep}})^{\mathcal{P}} \simeq \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep,\,colim}} \simeq \mathbf{Sup}^{\mathcal{V}}$$

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5.6.1 Presenting conically cocomplete $\mathcal{V}$-categories algebraically?

Consider moving from the presheaf-monad $\mathcal{P}$ on $\mathcal{V}$-**Cat**:

$$\mathcal{P} : \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}, \quad X \longmapsto [X^{\mathrm{op}}, \mathcal{V}], \quad \mathcal{P}X(\sigma, \tau) = \bigwedge_{z \in X} [\sigma z, \tau z]$$

to the Hausdorff submonad $\mathcal{H}$ via

$$j_X : \mathcal{H}X = \{A \mid A \subseteq X\} \longrightarrow \mathcal{P}X, \quad A \longmapsto (z \mapsto X(z, A) = \bigvee_{x \in A} X(z, x)).$$

where $\mathcal{H}X$ carries the initial (= cartesian) structure inherited from $\mathcal{P}X$ via j_X:

$$\mathcal{H}X(A, B) = \bigwedge_{z \in X} [\bigvee_{x \in A} X(z, x), \bigvee_{y \in B} X(z, y)] = \ldots = \bigwedge_{x \in A} \bigvee_{y \in B} X(x, y).$$

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$$\mathcal{H}X(A,B)=\bigwedge_{z\in X}[\bigvee_{x\in A}X(z,x),\bigvee_{y\in B}X(z,y)]=\ldots=\bigwedge_{x\in A}\bigvee_{y\in B}X(x,y).$$

5.6.2 Algebraic presentation of conically cocomplete $\mathcal{V}$-categories

Theorem (Akhvlediani-Clementino-T 2009, Stubbe 2009)

*Just like $\mathcal{P}$, also $\mathcal{H}$ becomes a lax-idempotent monad of the 2-category $\mathcal{V}$-**Cat**, lifting the power-set monad of **Set**, and making $j : \mathcal{H} \longrightarrow \mathcal{P}$ a monad morphism, which induces the forgetful functor*

$$(\mathcal{V}\text{-}\mathbf{Cat})^{\mathcal{P}_{\simeq}} \simeq \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{colim}} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{consup}} \simeq (\mathcal{V}\text{-}\mathbf{Cat})^{\mathcal{H}_{\simeq}},$$

$\mathcal{V}$-**Cat**$_{\mathrm{colim}}$*:*
(co)complete (= all weighted (co)limts exist) $\mathcal{V}$-categories, with cocontinous $\mathcal{V}$-functors;

$\mathcal{V}$-**Cat**$_{\mathrm{consup}}$:
conically cocomplete (= sups exist, Yoneda preserves) $\mathcal{V}$-cats, with sup-preserving $\mathcal{V}$-funs

5.7 $\mathcal{V}$-**Cat**$_{\mathrm{sep,colim}}$ as a quantification of **Sup**?

- Monadicity:

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep,colim}} \arrow[r, "\mathrm{monadic}"] \arrow[rrr, bend left, "\mathrm{monadic}"] & \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep}} \arrow[r, "\mathrm{monadic}"] & \mathcal{V}\text{-}\mathbf{Cat} \arrow[r, "\mathrm{topological}"] & \mathbf{Set}
\end{tikzcd}$$

- Self-duality:

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep,colim}} \arrow[r, "\cong"] & (\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep,colim}})^{\mathrm{op}}
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[d, "f \dashv"] \arrow[rr, mapsto] & & X^{\mathrm{op}} \\
Y \arrow[u, bend right, "f_*"'] \arrow[rr, mapsto] & & Y^{\mathrm{op}} \arrow[u, "(f_*)^{\mathrm{op}}"']
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- Symmetric monoidal-closed?

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- Self-duality: $\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep,colim}} \xrightarrow{\cong} (\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep,colim}})^{\mathrm{op}}$

$$\begin{tikzcd}
X \arrow[d, "f\ \dashv"] \arrow[rr, mapsto] & & X^{\mathrm{op}} \\
Y \arrow[u, bend right=60, "f_*"'] \arrow[rr, mapsto] & & Y^{\mathrm{op}} \arrow[u, "(f_*)^{\mathrm{op}}"']
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5.7 $\mathcal{V}$-**Cat**$_{\mathrm{sep,colim}}$ as a quantification of **Sup**?

- Monadicity:

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep,colim}} \arrow[r, "\text{monadic}"] \arrow[rrr, bend left=20, "\text{monadic}"] & \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep}} \arrow[r, "\text{monadic}"] & \mathcal{V}\text{-}\mathbf{Cat} \arrow[r, "\text{topological}"] & \mathbf{Set}
\end{tikzcd}$$

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\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[d, phantom, "\dashv", shift left=3] & \longmapsto & X^{\mathrm{op}} \\
Y \arrow[u, bend right, "f_*"'] & \longmapsto & Y^{\mathrm{op}} \arrow[u, "(f_*)^{\mathrm{op}}"']
\end{tikzcd}$$

- Symmetric monoidal-closed?

5.8 $\mathcal{V}$-**Cat**$_{\mathrm{sep,colim}}$ is symmetric monoidal closed

Having an equational presentation of separated cocomplete $\mathcal{V}$-categories, we construct the tensor product classifying "bimorphisms" in a standard manner:

Given objects X, Y, form the free object $\mathcal{P}_{\mathrm{d}}(X \times Y)$ (with the $\mathcal{V}$-powerset monad of **Set**) and then put

$$X \boxtimes Y = \mathcal{P}_{\mathrm{d}}(X \times Y)/\sim$$

with the least congruence relation $\sim$ making the Yoneda map $\mathbf{y} : X \times Y \longrightarrow \mathcal{P}_{\mathrm{d}}(X \times Y)/\sim$ a bimorphism; so, $\sim$ is generated by:

$$\mathbf{y}(u \odot x, y) \sim u \odot \mathbf{y}(x, y) \sim \mathbf{y}(x, u \odot y),$$

$$\mathbf{y}(\bigvee_{i \in I} x_i, y) \sim \bigvee_{i \in I} \mathbf{y}(x_i, y), \qquad \mathbf{y}(x, \bigvee_{i \in I} y_i) \sim \bigvee_{i \in I} \mathbf{y}(x, y_i)$$

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6.1 Cauchy sequences

$s = (x_n)_{n\in\mathbb{N}}$ sequence in $X \in \mathcal{V}$-**Cat**, $x \in X$

$$\mathrm{Cauchy}(s) := \bigvee_{N\in\mathrm{N}} \bigwedge_{m,n\geq N} X(x_m, x_n)$$

$$s \text{ is Cauchy} :\iff \mathrm{k} \leq \mathrm{Cauchy}(s)$$

$$\lambda_s(x) := \bigvee_{N\in\mathrm{N}} \bigwedge_{n\geq N} X(x_n, x) \quad (\text{“left-convergence value of } s \rightsquigarrow x\text{”})$$

$$\rho_s(x) := \bigvee_{N\in\mathrm{N}} \bigwedge_{n\geq N} X(x, x_n) \quad (\text{“right-convergence value of } s \rightsquigarrow x\text{”})$$

Facts:

$$E \overset{\lambda_s}{-\!\!\circ\!\!\to} X\,, \quad X \overset{\rho_s}{-\!\!\circ\!\!\to} E\,, \quad \text{with} \quad \lambda_s \cdot \rho_s \leq 1_X^*$$

$$s \text{ Cauchy} \iff 1_E^* \leq \rho_s \cdot \lambda_s \iff \lambda_s \dashv \rho_s$$

Definitions:

$$s \rightsquigarrow x :\iff \mathrm{k} \leq \bigvee_{N\in\mathrm{N}}\Big(\bigwedge_{m\geq N} X(x_m, x) \otimes \bigwedge_{n\geq N} X(x, x_n)\Big) \iff \mathrm{k} \leq \lambda_s(x) \otimes \rho_s(x)$$

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$$s \text{ Cauchy} \iff 1_E^* \leq \rho_s \cdot \lambda_s \iff \lambda_s \dashv \rho_s$$

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Facts:

$$\begin{tikzcd} E \arrow[r, "\lambda_s", "\circ" marking] & X, \quad X \arrow[r, "\rho_s", "\circ" marking] & E, \quad \text{with } \lambda_s \cdot \rho_s \leq 1_X^* \end{tikzcd}$$

$$s \text{ Cauchy} \iff 1_E^* \leq \rho_s \cdot \lambda_s \iff \lambda_s \dashv \rho_s$$

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Facts:

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X Cauchy-complete $:\iff$ every Cauchy sequence s in X converges to some point $x \in X$

6.2 Lawvere completeness

X Lawvere-complete : $\iff \forall\, \varphi \dashv \psi : X \relbar\!\!\circ\!\!\rightarrow W \;\; \exists\, f : W \longrightarrow X : \varphi = f_*,\ \psi = f^*$

$\iff \forall\, \varphi \dashv \psi : X \relbar\!\!\circ\!\!\rightarrow E \;\; \exists\, x \in X : \varphi = x_*,\ \psi = x^*$

$\Longrightarrow$ X Cauchy-complete

Conversely?

Auxiliary conditions on $\mathcal{V}$:

$\mathcal{V}$ integral $(\mathrm{k} = \top)$ and $\exists\, (\varepsilon_n)_{n\in\mathrm{N}}$ in $\mathcal{V}$: 1. $\varepsilon_n \leq \varepsilon_{n+1}$, 2. $\varepsilon_n \ll \mathrm{k}$, 3. $\bigvee_{n\in\mathrm{N}} \varepsilon_n = \mathrm{k}$

Then: $\forall\, \varphi \dashv \psi \; \exists\, s$ Cauchy in X : $\varphi = \lambda_s,\ \psi = \rho_s$

Theorem (Hofmann-Reis 2018)

If $\mathcal{V}$ satisfies the auxiliary conditions: X *Lawvere-complete* $\iff$ X *Cauchy-complete*

6.2 Lawvere completeness

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6.2 Lawvere completeness

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X \text{ Lawvere-complete :} \quad &\iff \forall\, \varphi \dashv \psi : X \mathrel{-\!\circ\!\!\to} W \;\; \exists\, f : W \longrightarrow X : \varphi = f_*,\ \psi = f^* \\
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6.2 Lawvere completeness

$$X \text{ Lawvere-complete} : \iff \forall\, \varphi \dashv \psi : X \longrightarrow\!\!\circ\!\!\rightarrow W \;\; \exists\, f : W \longrightarrow X : \varphi = f_*,\; \psi = f^*$$
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Theorem (Hofmann-Reis 2018)

If $\mathcal{V}$ satisfies the auxiliary conditions: X Lawvere-complete $\iff$ X Cauchy-complete

6.3 Cocompletion wrt a given class Φ of weights: conditions on Φ

W1 $f^* \in \Phi$, for every $\mathcal{V}$-functor f;

W2 $f^* \cdot \psi,\ \psi \cdot g^*,\ \psi \cdot h_* \in \Phi$, for all $\psi \in \Phi$ and $\mathcal{V}$-functors f, g, h with $h_* \in \Phi$, provided that the composites are defined;

W3 if $Y \xrightarrow{\psi} X$ satisfies $x^* \cdot \psi \in \Phi$ for all $x \in X$, then $\psi \in \Phi$;

W4 $f_* \in \Phi$, for every surjective $\mathcal{V}$-functor f.

Φ *cocompletion class* $:\iff$ (W1-3) hold; Φ *monadic cocompl. class* $:\iff$ (W1-4) hold.

Largest cocompletion class: all $\mathcal{V}$-distributors; trivially, it is monadic.
Least cocompletion class: $\{f^* \mid f\ \mathcal{V}\text{-functor}\}$; it may obviously fail to be monadic.
Lawvere cocompletion class: $\{\psi \mid \psi \text{ right adjoint}\}$; it fails to be monadic already for $\mathcal{V} = 2$.

$X \in \mathcal{V}$-**Cat** is Φ-*cocomplete* $:\iff$ all colimits of diagrams in X with weights in Φ exist.

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6.4 Cocompletion wrt a given class Φ of weights: pseudo-Φ-injectivity

For a cocompletion class Φ call

$f : X \to Y$ *Φ-dense* $:\iff f_* \in \Phi$;

X *pseudo-Φ-injective* $:\iff X$ pseudo-injective wrt fully faithful Φ-dense $\mathcal{V}$-functors;

Put

$$\begin{tikzcd}X \arrow[r, "\mathbf{y}_X^\Phi"'] \arrow[rr, bend left=10, "\mathbf{y}_X"] & \Phi X := \{\psi \in \mathcal{P}X \mid \psi \in \Phi\} \arrow[r, tail, "\mathrm{inc}_X^\Phi"'] & \mathcal{P}X\end{tikzcd}$$

Check:

- f has a right adjoint $\Longrightarrow f$ Φ-dense;
- f and $g : Y \to Z$ Φ-dense $\Longrightarrow g \cdot f$ Φ-dense;
- $g \cdot f$ Φ-dense and $f_* \cdot f^* = 1_Y^* \Longrightarrow g$ Φ-dense;
- $g \cdot f$ Φ-dense and g fully faithful $\Longrightarrow f$ Φ-dense;
- $\mathbf{y}_X^\Phi$ is Φ-dense;
- $(Y \overset{\psi}{\longrightarrow\!\!\!\!\circ}\, X) \in \Phi \iff$ the mate $\psi^\sharp : X \to \mathcal{P}Y$ factors through inc_X^Φ.

6.4 Cocompletion wrt a given class Φ of weights: pseudo-Φ-injectivity

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6.4 Cocompletion wrt a given class Φ of weights: pseudo-Φ-injectivity

For a cocompletion class Φ call

$f : X \to Y$ Φ-dense $:\iff$ $f_* \in \Phi$;

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6.4 Cocompletion wrt a given class Φ of weights: pseudo-Φ-injectivity

For a cocompletion class Φ call

$$\begin{array}{lcl} \textit{$f : X \to Y$ Φ-dense} & :\iff & f_* \in \Phi; \\ \textit{X pseudo-Φ-injective} & :\iff & X \text{ pseudo-injective wrt fully faithful } \Phi\text{-dense } \mathcal{V}\text{-functors;} \end{array}$$

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- $g \cdot f$ Φ-dense and $f_* \cdot f^* = 1_Y^*$ $\Longrightarrow$ g Φ-dense;
- $g \cdot f$ Φ-dense and g fully faithful $\Longrightarrow$ f Φ-dense;
- $\mathbf{y}_X^{\Phi}$ is Φ-dense;
- $(Y \overset{\psi}{\longrightarrow\!\!\!\circ\!\!\!\rightarrow} X) \in \Phi \iff$ the mate $\psi^\sharp : X \to \mathcal{P}Y$ factors through inc_X^{Φ}.

6.4 Cocompletion wrt a given class Φ of weights: pseudo-Φ-injectivity

For a cocompletion class Φ call

$f : X \to Y$ Φ-dense $\quad :\iff \quad f_* \in \Phi$;
X pseudo-Φ-injective $\quad :\iff \quad X$ pseudo-injective wrt fully faithful Φ-dense $\mathcal{V}$-functors;

Put

$$\begin{tikzcd} X \arrow[r, "\mathbf{y}_X^{\Phi}"'] \arrow[rr, bend left=15, "\mathbf{y}_X"] & \Phi X := \{\psi \in \mathcal{P}X \mid \psi \in \Phi\} \arrow[r, tail, "\mathrm{inc}_X^{\Phi}"'] & \mathcal{P}X \end{tikzcd}$$

Check:

- f has a right adjoint $\Longrightarrow$ f Φ-dense;
- f and $g : Y \to Z$ Φ-dense $\Longrightarrow$ $g \cdot f$ Φ-dense;
- $g \cdot f$ Φ-dense and $f_* \cdot f^* = 1_Y^*$ $\Longrightarrow$ g Φ-dense;
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6.5 Cocompletion wrt a given class Φ of weights: theorems

Theorem (Clementino-Hofmann 2009)

Let Φ be a cocompletion class.

- *The following properties for a $\mathcal{V}$-category X are equivalent:*
  - (i) *X is Φ-cocomplete,* i.e. *X has all colimits with weights in Φ;*
  - (ii) *X carries the structure of a pseudo-algebra with respect to the Φ-presheaf monad $(\Phi, \mathbf{s}^{\Phi}, \mathbf{y}^{\Phi})$ on $\mathcal{V}$-**Cat**;*
  - (iii) *the Yoneda $\mathcal{V}$-functor $\mathbf{y}^{\Phi}_X$ has a pseudo-retraction; that is: there is a $\mathcal{V}$-functor $h : \mathcal{P}^{\Phi} X \to X$ with $h\,\mathbf{y}^{\Phi}_X \simeq 1_X$;*
  - (iv) *X is pseudo-Φ-injective in $\mathcal{V}$-**Cat**.*
- $\Phi \dashv (\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep},\Phi\text{-colim}} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat})$.
- *If Φ is monadic, then $\mathcal{V}$-**Cat**$_{\mathrm{sep},\Phi\text{-colim}}$ is monadic over* **Set**.

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6.6 Cauchy completion of a $\mathcal{V}$-category à la Lawvere

Let $\mathcal{V}$ satisfy $\mathrm{k} = \top$ and $\exists\,(\varepsilon_n)_{n\in\mathrm{N}}$ in $\mathcal{V}$: 1. $\varepsilon_n \leq \varepsilon_{n+1}$, 2. $\varepsilon_n \ll \mathrm{k}$, 3. $\bigvee_{n\in\mathrm{N}} \varepsilon_n = \mathrm{k}$;

Consider $\Phi := \{\psi \mid \psi$ right adjoint $\mathcal{V}$-distributor$\}$, and let $X \in \mathcal{V}$-**Cat**. Then

$$\Phi X = \{\psi \in \mathcal{P}_{\mathcal{V}}X \mid \psi \text{ right adjoint}\} = \{\rho_s \mid s = (x_n)_n \text{ Cauchy sequence in } X\}$$

with $\rho_s(x) = \bigvee_{N\in\mathrm{N}} \bigwedge_{n\geq N} X(x, x_n)$ $(x \in X)$, and

1. (trivially) $(s \sim s' \iff \rho_s = \rho_{s'})$ is an equivalence relation on the set of all Cauchy sequences in X, with projection $s \mapsto \rho_s$;
2. ΦX is Cauchy complete;
3. the restricted Yoneda $\mathcal{V}$-functor $X \to \Phi X,\ y \mapsto \rho_{(y)_n},$ is a reflection of X into the full subcategory of Cauchy complete $\mathcal{V}$-categories.

7.1 The category $\mathbf{Set}/\!/\mathcal{V} = \mathcal{V}$-**wSet** of $\mathcal{V}$-weighted or -normed sets

- Defining $\mathbf{Set}/\!/\mathcal{V}$:

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "|-|_A"'] & \arrow[d, phantom, "\leq"] & B \arrow[dl, "|-|_B"] \\ & \mathcal{V} & \end{tikzcd} \iff \forall\, a \in A: \; |a|_A \leq |\varphi a|_B$$

- $\mathbf{Set}/\!/\mathcal{V}$ is topological over $\mathbf{Set}$:

$$\begin{tikzcd} (\, A \arrow[rr, "\varphi_i"] \arrow[dr, "|-|"'] & \arrow[d, phantom, "\leq"] & B_i\,)_{i\in I} \arrow[dl, "|-|"] \\ & \mathcal{V} & \end{tikzcd} \text{ initial} \iff |a| = \bigwedge_{i\in I} |\varphi_i a|$$

- $\mathbf{Set}/\!/\mathcal{V}$ is symmetric monoidal-closed:

$$A \otimes B = (\, A \times B,\; |(a,b)| = |a| \otimes |b|\,), \qquad E = (1 = \{*\},\; |*| = \mathrm{k})$$

$$[A,B] = (\, \mathbf{Set}(A,B),\; |\varphi| = \bigwedge_{a\in A} [|a|, |\varphi a|]\,)$$

7.2 The category **Cat**$//\mathcal{V}$ = $\mathcal{V}$-**wCat** of (small) $\mathcal{V}$-weighted categories

Objects of **Cat**$//\mathcal{V}$ are (small) categories $\mathbb{X}$ enriched in **Set**$//\mathcal{V}$; this means (neglecting $\forall$):

$$\mathbb{X}(x,y) \otimes \mathbb{X}(y,z) \longrightarrow \mathbb{X}(x,z) \quad \text{and} \quad E \longrightarrow \mathbb{X}(x,x) \qquad \text{live in } \mathbf{Set}//\mathcal{V}$$

$$\Longleftrightarrow \quad |f| \otimes |g| = |(f,g)| \leq |g \cdot f| \qquad \text{and} \quad \mathrm{k} \leq |1_x|$$

$$\Longleftrightarrow \quad |\text{-}| : \mathbb{X} \longrightarrow (\mathcal{V}, \otimes, \mathrm{k}) \text{ is a lax functor}$$

For a functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ to be enriched in **Set**$//\mathcal{V}$ means (without universal quantifiers):

$$\mathbb{X}(x,y) \longrightarrow \mathbb{Y}(Fx, Fy) \quad \text{lives in } \mathbf{Set}//\mathcal{V}$$

$$\Longleftrightarrow \quad |f| \leq |Ff|$$

$$\Longleftrightarrow \quad \begin{tikzcd} \mathbb{X} \arrow[rr, "F"] \arrow[dr, "|\text{-}|"'] & \leq & \mathbb{Y} \arrow[dl, "|\text{-}|"] \\ & \mathcal{V} & \end{tikzcd}$$

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7.3 The adjunction $\mathrm{s}\dashv\mathrm{i}$, monoidal-closed structure, preserved by i, s

$\mathcal{V}$-**Cat**		**Cat**$//\mathcal{V}$
$X, \quad X(x,y)\otimes X(y,z)\le X(x,z)$ $\mathrm{k}\le X(x,x)$	$\xmapsto{\ \mathrm{i}\ }$	$\mathrm{i}X, \quad \mathrm{ob}(\mathrm{i}X)=X$ $x\xrightarrow{(x,y)}y\,, \quad \lvert(x,y)\rvert = X(x,y)$
$\mathrm{s}\mathbb{X}=\mathrm{ob}\mathbb{X}, \quad \mathrm{s}\mathbb{X}(x,y)=\bigvee\{\lvert f\rvert \mid f: x\to y\}$	$\xleftarrow{\ \mathrm{s}\ }\!\!\dashv$	$\mathbb{X}, \quad \lvert f\rvert\otimes\lvert g\rvert\le\lvert g\cdot f\rvert$ $\mathrm{k}\le\lvert 1_x\rvert$
$X\otimes Y = X\times Y$ (as a set) $(X\otimes Y)((x,y),(x',y')) = X(x,x')\otimes Y(y,y')$		$\mathbb{X}\otimes\mathbb{Y}=\mathbb{X}\times\mathbb{Y}$ (as a category) $\lvert(f,g)\rvert = \lvert f\rvert\otimes\lvert g\rvert$
$[X,Y] = \mathcal{V}\text{-}\mathbf{Cat}(X,Y)$ (as a set) $[X,Y](f,g) = \bigwedge_{x\in X} Y(fx,gx)$		$[\mathbb{X},\mathbb{Y}] = (\mathbf{Cat}//\mathcal{V})(\mathbb{X},\mathbb{Y})$ (as a cat) $\lvert\, F\xrightarrow{\alpha}G\,\rvert = \bigwedge_{x\in\mathrm{ob}\mathbb{X}}\lvert\alpha_x\rvert$

7.4.1 Example: $(\mathcal{V}, \leq, \otimes, \mathrm{k}) = (2, \bot < \top, \wedge, \top)$

2-**Cat** = **Ord**

Cat//2 = **sCat**

$$X, \quad x \leq y \ \wedge \ y \leq z \Longrightarrow x \leq z \qquad \xmapsto{\ \mathrm{i}\ } \qquad \mathrm{i}X, \quad \mathrm{ob}(\mathrm{i}X) = X$$

$$\top \Longrightarrow x \leq x \qquad\qquad (\, x \xrightarrow{(x,y)} y) \in \mathcal{S} \Longleftrightarrow x \leq y$$

$$\mathrm{s}\mathbb{X} = \mathrm{ob}\mathbb{X}, \quad x \leq y \Longleftrightarrow \exists (f : x \to y) \in \mathcal{S} \qquad \xleftarrow{\ \mathrm{s}\ } \!\!\dashv \qquad \mathbb{X}, \mathcal{S}, \quad f, g \in \mathcal{S} \Longrightarrow g \cdot f \in \mathcal{S}$$

$$\top \Longrightarrow 1_x \in \mathcal{S}$$

$$X \otimes Y = X \times Y \qquad\qquad \mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y} \text{ (as a category)}$$

$$(x, y) \leq (x', y') \Longleftrightarrow \ x \leq x' \ \wedge \ y \leq y' \qquad\qquad \mathcal{S}_{\mathbb{X} \otimes \mathbb{Y}} = \mathcal{S}_{\mathbb{X}} \times \mathcal{S}_{\mathbb{Y}}$$

$$[X, Y] = \mathbf{Ord}(X, Y) \qquad\qquad [\mathbb{X}, \mathbb{Y}] = \mathbf{sCat}(\mathbb{X}, \mathbb{Y}) \text{ (as a cat)}$$

$$f \leq g \Longleftrightarrow \forall x \in X : fx \leq gx \qquad\qquad \alpha \in \mathcal{S}_{[\mathbb{X}, \mathbb{Y}]} \Longleftrightarrow \forall x \in \mathrm{ob}\mathbb{X} : \alpha_x \in \mathcal{S}_{\mathbb{Y}}$$

7.4.2 Example: $(\mathcal{V}, \leq, \otimes, \mathrm{k}) = ([0,\infty], \geq, +, 0)$

$[0,\infty]$-**Cat** $=$ **Met** $\qquad$ **Cat**$//[0,\infty]$ $=$ **wCat**

$$X, \quad d(x,y) + d(y,z) \geq d(x,z) \qquad \xmapsto{\mathrm{i}} \qquad \mathrm{i}X, \quad \mathrm{ob}(\mathrm{i}X) = X$$

$$0 \geq d(x,x) \qquad\qquad x \xrightarrow{(x,y)} y\ , \quad |(x,y)| = d(x,y)$$

$$\mathrm{s}\mathbb{X} = \mathrm{ob}\mathbb{X}, \quad d(x,y) = \inf_{f:x\to y} |f| \qquad \overset{\mathrm{s}}{\longmapsfrom} \qquad \mathbb{X}, \quad |f| + |g| \geq |g \cdot f|$$

$$0 \geq |1_x|$$

$$X \otimes Y = X \times Y \qquad\qquad \mathbb{X} \otimes \mathbb{Y} = \mathbb{X} \times \mathbb{Y} \text{ (as a category)}$$

$$d((x,y),(x',y')) = d(x,y) + d(y,y') \qquad |(f,g)| = |f| + |g|$$

$$[X,Y] = \mathbf{Met}(X,Y) \qquad\qquad [\mathbb{X},\mathbb{Y}] = \mathbf{wCat}(\mathbb{X},\mathbb{Y}) \text{ (as a cat)}$$

$$d(f,g) = \sup_{x\in X} d(fx,gx) \qquad\qquad |\ F \xrightarrow{\alpha} G\ | = \sup_{x\in\mathrm{ob}\mathbb{X}} |\alpha_x|$$

7.5.1 Some elementary examples of weighted categories, I

We saw:

$\mathcal{V}$-categories (and their functors) are $\mathcal{V}$-weighted categories (and their functors); in fact, they are precisely the $\mathcal{V}$ weighted categories with indiscrete underlying category.

Question: May **Set** be "naturally" $[0, \infty]$-weighted?

Goal 1: Let $|f|$ measure the degree to which a map $f : X \to Y$ fails to be surjective.

Simply put $|f| := \#(Y \setminus f(X)) \in \mathbb{N} \cup \{\infty\} \subseteq [0, \infty]$.

Then: $0 \geq |\mathrm{id}_X|$, and with $g : Y \to Z$ we have $|f| + |g| \geq |g \cdot f|$

since (assuming Choice and $Y \cap Z = \emptyset$) there is an injective map

$$Z \setminus (g(f(X))) \longrightarrow (Y \setminus f(X)) + (Z \setminus g(Y)).$$

Note: f surjective $\iff |f| = 0$.

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7.5.2 Some elementary examples of weighted categories, II

Question: May something similar be done for injectivity? That is:

Goal 2: Let $|f|$ measure the degree to which a map $f : X \to Y$ fails to be injective.

First consider $\# f := \sup_{y \in Y} \# f^{-1}y$; then, with $g : Y \to Z$, we have:

$$\# g \cdot \# f = (\sup_{z \in Z} \# g^{-1}z) \cdot (\sup_{y \in Y} \# f^{-1}y) \geq \sup_{z \in Z} \#(\bigcup_{y \in g^{-1}z} f^{-1}y) = \#(g \cdot f), \quad 1 \geq \# \mathrm{id}_X$$

Not what we wanted! But $([1,\infty], \geq, \cdot, 1) \xrightarrow[\log]{\cong} ([0,\infty], \geq, +, 0)$ comes to the rescue:

Put $|f| := \max\{0, \log \# f\}$; then: $|g| + |f| \geq |g \cdot |f|$. $0 \geq |\mathrm{id}_X|$.

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7.5.3 A more interesting example of a (large) weighted category: **Lip**

$\mathrm{ob}\,\mathbf{Lip} = \mathrm{ob}\,\mathbf{Met}$, $\quad \mathbf{Lip}(X, Y) = \mathbf{Set}(X, Y)$; $\quad$ why call this category **Lip** ??

Recall: $f : X \to Y$ is $K(\geq 0)$-Lipschitz $\iff \forall x \neq x' : d(fx, fx') \leq K\, d(x, x')$

In particular: $f : X \to Y$ is a morphism in **Met** $\iff$ f is 1-Lipschitz

Question: How far is an arbitrary map f away from being 1-Lipschitz?

Answer: Find the least Lipschitz constant $K \geq 1$ for f (admitting $K = \infty$)

That is: $\mathrm{Lip}(f) = \max\{1, \sup_{x \neq x'} \dfrac{d(fx, fx')}{d(x, x')}\}$ (assuming temporarily that X be separated)

Then: $\mathrm{Lip}(g) \cdot \mathrm{Lip}(f) \geq \mathrm{Lip}(g \cdot f), \quad 1 \geq \mathrm{Lip}(\mathrm{id}_X)$

No problem:

$$([1, \infty], \geq, \cdot, 1) \xrightarrow[\log]{\cong} ([0, \infty], \geq, +, 0)\ , \quad |f| = \max\{0, \sup_{x, x'} (\log d(fx, fx') - \log d(x, x'))\}$$

Then: $|g| + |f| \geq |g \cdot f|, \quad 0 \geq |\mathrm{id}_X|, \qquad$ (f 1-Lipschitz $\iff |f| = 0$)

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Question: How far is an arbitrary map f away from being 1-Lipschitz?

Answer: Find the least Lipschitz constant $K \geq 1$ for f (admitting $K = \infty$)

That is: $\mathrm{Lip}(f) = \max\{1, \sup\limits_{x \neq x'} \dfrac{d(fx, fx')}{d(x, x')}\}$ (assuming temporarily that X be separated)

Then: $\mathrm{Lip}(g) \cdot \mathrm{Lip}(f) \geq \mathrm{Lip}(g \cdot f), \quad 1 \geq \mathrm{Lip}(\mathrm{id}_X)$

No problem:

$$([1,\infty], \geq, \cdot, 1) \xrightarrow[\log]{\cong} ([0,\infty], \geq, +, 0)\ , \quad |f| = \max\{0, \sup_{x,x'} (\log d(fx, fx') - \log d(x, x'))\}$$

Then: $|g| + |f| \geq |g \cdot f|, \quad 0 \geq |\mathrm{id}_X|, \qquad (f \text{ 1-Lipschitz} \iff |f| = 0)$

7.5.3 A more interesting example of a (large) weighted category: **Lip**

$\mathrm{ob}\,\mathbf{Lip} = \mathrm{ob}\,\mathbf{Met}$, $\quad \mathbf{Lip}(X,Y) = \mathbf{Set}(X,Y)$; why call this category **Lip** ??

Recall: $\quad f : X \to Y$ is $K(\geq 0)$-Lipschitz $\iff \forall x \neq x' : d(fx, fx') \leq K\, d(x,x')$

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7.6 On the axiomatics for weighted/normed categories

The category $\mathbb{X}$ is $\mathcal{V}$-weighted by $|\text{-}| : \mathbb{X} \longrightarrow \mathcal{V}$ if

$$\begin{array}{lll}
\mathrm{k} \le |1_x| & & \\
|g| \otimes |f| \le |g \cdot f| & \iff |f| \le \bigwedge\limits_{g} [|g|, |g \cdot f|] & \iff |f| = \bigwedge\limits_{g} [|g|, |g \cdot f|] \\
 & \iff |g| \le \bigwedge\limits_{f} [|f|, |g \cdot f|] & \iff |g| = \bigwedge\limits_{f} [|f|, |g \cdot f|]
\end{array}$$

The $\mathcal{V}$-weighted category $\mathbb{X}$ is *right/left cancellable* if

$$|f| \otimes |g \cdot f| \le |g| \iff |f| \le \bigwedge\limits_{g} [|g \cdot f|, |g|] =: |f|^{\mathrm{R}}$$ (right cancellable)

$$|g| \otimes |g \cdot f| \le |f| \iff |g| \le \bigwedge\limits_{f} [|g \cdot f|, |f|] =: |g|^{\mathrm{L}}$$ (left cancellable; Kubiś: “norm”)

Facts (Insall-Luckhardt for $\mathcal{V} = [0, \infty]$): $\mathbb{X}$ weighted by $|\text{-}| \Longrightarrow \mathbb{X}$ weighted by $|\text{-}|^{\mathrm{R}}$ and $|\text{-}|^{\mathrm{L}}$, and $|f| \le |f|^{\mathrm{RR}}$, $|f| \le |f|^{\mathrm{LL}}$.

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7.7 The underlying ordinary category $\mathbb{X}_0$ of a $\mathcal{V}$-weighted category $\mathbb{X}$

Note:

An isomorphism f in $\mathbb{X}$ may not satisfy $\mathrm{k} \leq |f|$, and even when it does, we may not have $\mathrm{k} \leq |f^{-1}|$ (unless the weight is left/right cancellable). Still, in many of the examples with $\mathcal{V} = [0, \infty]$ considered in the literature, morphisms f, and especially isomorphisms, of norm 0 play an important role. They are called “modulators” by Insall-Luckhardt.

Question:

What is the “enriched significance” of considering morphisms f with $\mathrm{k} \leq |f|$?

Answer:

These are precisely the morphisms of the underlying ordinary category $\mathbb{X}_0$ of the ($\mathbf{Set}//\mathcal{V}$)-enriched category $\mathbb{X}$.

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7.8.1 $\mathcal{V}$-weighted cats *vs.* $\mathcal{V}$-metrically enriched cats: syntax prep

Recall: groups $(X, -, 0)$ in subtractive notation:

$$x - 0 = x,\ x - x = 0,\ (x - y) - (z - y) = x - z$$

Write $\mathcal{V}$-**Met** for $\mathcal{V}$-**Cat**$_{\mathrm{sym}}$: "$\mathcal{V}$-metric spaces" = $\mathcal{V}$-categories X with $X(x, y) = X(y, x)$

Form the category $\mathcal{V}$-**MetGrp** of "$\mathcal{V}$-metric groups":

objects are $\mathcal{V}$-metric spaces X with a group structure that makes distances invariant under translations:

$$X(x, y) = X(x - z, y - z);$$

morphisms are $\mathcal{V}$-contractive homomorphisms.

$\mathcal{V}$-**MetGrp** inherits its symmetric monoidal structure from $\mathcal{V}$-**Cat** and the cartesian cat **Grp**.

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7.8.2 $\mathcal{V}$-metric groups as $\mathcal{V}$-weighted groups

The category $\mathbf{Grp}/\!/\mathcal{V}$ has as

objects: $\mathcal{V}$-weighted sets $(X, |\text{-}|)$ with a group structure such that

$$\mathrm{k} \leq |0|, \quad |x| \otimes |y| \leq |x - y|;$$

morphisms live in both, $\mathbf{Set}/\!/\mathcal{V}$ and $\mathbf{Grp}$.

Obtain:

$$\begin{tikzcd}
\mathbf{Grp}/\!/\mathcal{V} \arrow[r, leftrightarrow, "\cong"] & \mathcal{V}\text{-}\mathbf{MetGrp} \\
X \arrow[r, mapsto] & X(x,y) = |x - y| \\
|x| = X(x,0) & X \arrow[l, mapsto]
\end{tikzcd}$$

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Obtain:

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X \arrow[rr, mapsto] & & X(x,y) = |x - y| \\
|x| = X(x,0) & & X \arrow[ll, mapsto]
\end{tikzcd}$$

7.9 $\mathcal{V}$-weighted cats *vs* $\mathcal{V}$-metrically enriched cats *vs* $\mathcal{V}$-metagories

$$\begin{tikzcd}
 & \mathbf{Grp}/\!/\mathcal{V} \arrow[r, leftrightarrow, "\cong"] \arrow[d] & \mathcal{V}\text{-}\mathbf{MetGrp} \arrow[d] \\
\mathcal{V} \arrow[r, tail, "\mathrm{i}"] & \mathbf{Set}/\!/\mathcal{V} & \mathcal{V}\text{-}\mathbf{Met}
\end{tikzcd}$$

$$\begin{tikzcd}
 & (\mathbf{Grp}/\!/\mathcal{V})\text{-}\mathbf{Cat} \arrow[r, leftrightarrow, "\cong"] \arrow[d] & (\mathcal{V}\text{-}\mathbf{MetGrp})\text{-}\mathbf{Cat} \arrow[d] & \\
\mathcal{V}\text{-}\mathbf{Cat} \arrow[r, tail, "\mathrm{i}"] & (\mathbf{Set}/\!/\mathcal{V})\text{-}\mathbf{Cat} = \mathbf{Cat}/\!/\mathcal{V} & (\mathcal{V}\text{-}\mathbf{Met})\text{-}\mathbf{Cat} \arrow[r, tail] & \mathcal{V}\text{-}\mathbf{Metag}
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$$\begin{tikzcd}
 & \mathbf{Grp}/\!/\mathcal{V} \arrow[r, leftrightarrow, "\cong"] \arrow[d] & \mathcal{V}\text{-}\mathbf{MetGrp} \arrow[d] \\
\mathcal{V} \arrow[r, tail, "\mathrm{i}"] & \mathbf{Set}/\!/\mathcal{V} & \mathcal{V}\text{-}\mathbf{Met}
\end{tikzcd}$$

$$\begin{tikzcd}
 & (\mathbf{Grp}/\!/\mathcal{V})\text{-}\mathbf{Cat} \arrow[r, leftrightarrow, "\cong"] \arrow[d] & (\mathcal{V}\text{-}\mathbf{MetGrp})\text{-}\mathbf{Cat} \arrow[d] & \\
\mathcal{V}\text{-}\mathbf{Cat} \arrow[r, tail, "\mathrm{i}"] & (\mathbf{Set}/\!/\mathcal{V})\text{-}\mathbf{Cat} = \mathbf{Cat}/\!/\mathcal{V} & (\mathcal{V}\text{-}\mathbf{Met})\text{-}\mathbf{Cat} \arrow[r, tail] & \mathcal{V}\text{-}\mathbf{Metag}
\end{tikzcd}$$

7.10.1 Principal references, I

F. W. Lawvere: Metric spaces, generalized logic, and closed categories

Rendiconti del Seminario Matematico e Fisico di Milano 43:135–166, 1973.

Republished in *Reprints in Theory and Applications of Categories* 1, 2002.

(This paper not only introduces metric spaces as small categories enriched in the extended real half-line, considered as a symmetric monoidal-closed category under addition, but it is also the birthplace of *normed categories*, introduced as categories enriched in a certain symmetric monoidal category of *normed sets*.)

D. Hofmann, G.J. Seal, W.T. (eds.): *Monoidal Topology–A Categorical Approach to Order, Metric and Topology*, Cambridge University Press, 2014.

(This book studies the category $(\mathbb{T}, \mathcal{V})$-**Cat**, for a **Set**-monad $\mathbb{T}$ which is assumed to interact with the $\mathcal{V}$-presheaf monad $\mathcal{P}_{\mathcal{V}}$ via a lax distributive law; for $\mathbb{T}$ the identity monad on **Set**, one obtains the category $\mathcal{V}$-**Cat** as considered in these lectures.)

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