# Farness via Galois adjunctions and a separation theorem for uniform frames

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TACL

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- Complete lattices: $f : A \to B^{op}$ and $g : B \to A^{op}$ are complete join homomorphisms.

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$a \wedge x \leq y \Leftrightarrow x \leq a \to y$ (Heyting operator)

We have the pseducomplement of an element $a \in L$: $a^* = a \to 0$.

$$\mathrm{P} : \begin{array}{ccc} L & \longrightarrow & L \\ x & \longmapsto & x^* \end{array}$$

P is a self-dual Galois adjoint : $a \leq b^* \Leftrightarrow b \leq a^*$.

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  (r1) $(p, -) \wedge (-, q) = 0$ if $q \leq p$,
  (r2) $(p, -) \vee (-, q) = 1$ if $p < q$,
  (r3) $(p, -) = \bigvee_{r>p} (r, -)$,
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For rationals $p \leq q$, the element $(p, -) \wedge (-, q)$ in $\mathcal{L}(\mathbb{R})$ is denoted by $(p, q)$.
A **continuous real-valued function** on a frame $L$ is a frame homomorphism $\mathcal{L}(\mathbb{R}) \to L$.

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$U\left(\bigvee_{i \in I} x_i\right) = \bigvee_{i \in I} Ux_i$ : For every cover $U$ of $L$ we have an adjunction

$$\begin{array}{rccc} S_U : & L & \longrightarrow & L \\ & x & \longmapsto & Ux \end{array} \quad \dashv \quad \begin{array}{rccc} \widetilde{S_U} : & L & \longrightarrow & L \\ & y & \longmapsto & y/U = \bigvee \{b \mid Ub \leq y\} . \end{array}$$

# Uniform frames

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(U1) $U \in \mathcal{U}$ and $U \leq V$ implies $V \in \mathcal{U}$,

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(pre-)uniformity: (U1), (U2), (U3)
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A frame homomorphism $h\colon L \to M$ is a **uniform homomorphism**

$$h\colon (L, \mathcal{U}) \to (M, \mathcal{V})$$

if $h[U] \in V$ for every $U \in \mathcal{U}$.
for bases: For every $U \in \mathcal{U}$, $V \leq h[U]$ for some $V \in \mathcal{V}$.

# Metric Uniformity of $\mathcal{L}(\mathbb{R})$

For every $n$ natural

$$D_n = \left\{(p,q) \in \mathcal{L}(\mathbb{R}) \mid q - p = \tfrac{1}{n}\right\}$$

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A **uniform continuous real-valued function** on a (pre-)uniform frame $(L,\mathcal{U})$ is a frame homomorphism $f\colon \mathcal{L}(\mathbb{R}) \to L$ such that

$$\forall n \in \mathbb{N} \quad U \leq f[D_n] \text{ for some } U \in \mathcal{U}.$$

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Properties:

- $a$ and $b$ are $U$-far and $V \leq U \Rightarrow a$ and $b$ are $V$-far.
- $a$ and $b$ are $U$-far, $c \leq a$ and $d \leq b \Rightarrow c$ and $d$ are $U$-far.
- $a$ and $b$ are $U$-far $\Leftrightarrow a^{**}$ and $b^{**}$ are $U$-far.
- $a$ and $b$ are $U$-far for some $U \in \mathcal{U} \Leftrightarrow a \triangleleft_{\mathcal{U}} b^*$.
- $a$ and $b$ are $U$-far $\Rightarrow a^* \vee b^* = 1$.

# Characterization Uniformly continuous real-valued functions

**Theorem**

*Let* $(L,\mathcal{U})$ *be a (pre-)uniform frame and* $f : \mathcal{L}(\mathbb{R}) \to L$ *a frame homomorphism, then the following are equivalent:*

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(ii) *For every $\delta \in \mathbb{Q}^+$ there is $U \in \mathcal{U}$ such that $f(–,r)$ and $f(s,–)$ are $U$-far for $s, r \in \mathbb{Q}$ with $s - r > \frac{1}{\delta}$.*

# Farness + Galois Adjunctions

Let $U$ be a cover of $L$

$$\mathsf{P} : \begin{array}{ccc} L & \longrightarrow & L \\ a & \longmapsto & a^* \end{array}$$

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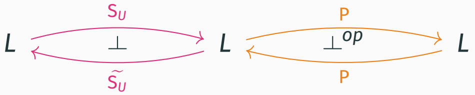

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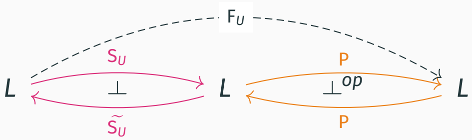


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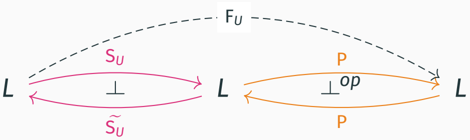


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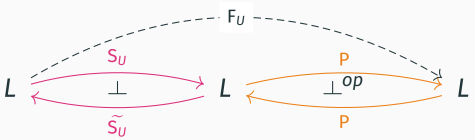


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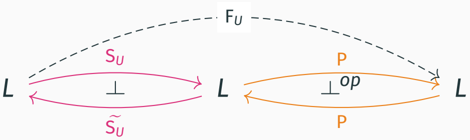


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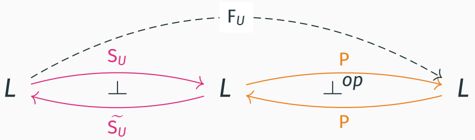


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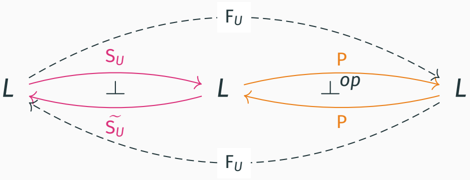


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**Theorem**

*Let* $(L,\mathcal{U})$ *be a (pre-)uniform frame. If a and b are U-far for some* $U \in \mathcal{U}$ *then there is a uniformly continuous* $f\colon \mathcal{L}(\mathbb{R}) \to L$ *with* $0 \le f \le 1$ *such that* $f(0,-) \wedge a = 0$ *and* $f(-,1) \wedge b = 0$.

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- Check relations (r1)-(r6).
- Check that $\forall\delta \in \mathbb{Q}^+$ there is $U \in \mathcal{U}$ such that $a_r$ and $b_s$ are $U$-far for every $s - r > \frac{1}{\delta}$ (uniformity).

# Construction

- We will build $\{a_r\}_{r\in\mathbb{D}} \subseteq L$ and $\{b_s\}_{s\in\mathbb{D}} \subseteq L$ where

$$\begin{aligned}\mathbb{D} &= \left\{ \tfrac{m}{2^n} \mid n = 1, 2, \cdots, m = 0, \cdots, 2^n \right\} \\ &= \{0, 1\} \cup \bigcup_{n\in\mathbb{N}} \left\{ \tfrac{2k-1}{2^n} \mid k = 1, \cdots 2^{n-1} \right\}.\end{aligned}$$

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|---|---|---|---|
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$$a_0 = a, a_1 = 1 \text{ and } a_{\frac{2k-1}{2^n}} = F_{U_n}\left(b_{\frac{k}{2^{n-1}}}\right) \text{ for } k = 1, \cdots, 2^n - 1$$

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# References


- A. B. Avilez and J. Picado, Uniform continuity of pointfree real functions via farness and related Galois connections, DMUC preprint 22-08.
- B. Banaschewski, *The real numbers in pointfree topology*, Textos de Matemática, Vol. 12, University of Coimbra (1997).
- J. Picado and A. Pultr, *Frames and locales: Topology without points*, Frontiers in Mathematics, vol. 28, Springer, Basel (2012).
- D. Preiss and J. Vilimovský, In-between theorems in uniform spaces, *Trans. Amer. Math. Soc.* 261 (1980) 483–501.
- A. Pultr, Pointless uniformities I, *Comment. Math. Univ. Carolin.* 25 (1984) 91–104.
- J. M. Smirnov, On proximity spaces, (Russian), *Mat. Sb.* 31 (1952) 543–574; English trans.: *Amer. Math. Soc. Transl.* (*Ser. 2*) 38 (1964) 5–35.

**Thank you!**