Modality in worlds with different logics

Alfredo Roque Freire

Universidade de Aveiro
CIDMA

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Problem

This is a joint work with Manuel António Martins - CIDMA University of Aveiro

In this presentation we would like to consider

1. non-trivial modal structures where worlds can operate in different logics.
2. the thesis that:
    1. (NAW) necessity means 'true in all accessed worlds' or the similar version.
    2. (PAW) possibility means 'true in some accessed world'.
3. modal structures where NAW and PAW fails.
4. and frames that study the dynamic between logic systems.

Quine's Charge

Can we change logic and still preserve meaning?
"change of logic, change of subject" [Qui86, p. 80]

Kripke-Barcan Rigid designation

Even though Newton discovered the theory of universal gravitation, the phrase

1. 'It is possible that Newton died in infancy'
2. 'It is possible that the one who discovered the theory of universal gravitation died in infancy.'

Have different truth values...

Lattice semantics

Most studied logic systems have a semantics where **conjunction** and **disjunction** are **meet** and **join** in the lattice of truth values (with the appropriate designated values).

1. Classical: boolean algebras.
2. Intuitionism: Heytin algebras.
3. Logic of Paradox: glut lattice.

Base lattice L

Definition

A world w over L is a pair $\langle A, i\rangle$ such that

1. A is a complete sublattice of L and it is closed under $-(x)$ operation.
2. and i is a name for the world.

Note that L need not be complete.

Definition

A model M over L is a tuple $\langle W, R, v\rangle$ where W is a collection of worlds over L, R is a relation in $W \times W$ and v is a valuation for propositional values (i.e. $v : Var \longrightarrow A_w$).

Valuation - in world

Definition

The valuation v_w of formulas in a world $w = \langle A, v \rangle$ is such that

1. $v_w(\neg\varphi) = -v_w(\varphi)$.
2. $v_w(\varphi \vee \psi) = v_w(\varphi) + v_w(\psi)$.
3. $v_w(\varphi \wedge \psi) = v_w(\varphi).v_w(\psi)$.
4. $v_w(\varphi \rightarrow \psi) = v_w(\varphi) \supset v_w(\psi)$

An important alternative for implication:

$$v_w(\varphi \rightarrow \psi) = \begin{cases} \bigcup A, \text{ if } v_w(\varphi) \leq v_w(\psi) \\ v_w(\psi), \text{otherwise} \end{cases}$$

Valuation - modal

Definition

In a base lattice L, a value $a \in L$ is interpreted in a sublattice L' as:

1. **Down-interpretation** $a^{L'} = \bigcup_{L'}\{x \in L' \mid x \leq a\}$.
2. **Up-interpretation** $a^{L'} = \bigcap_{L'}\{x \in L' \mid x \geq a\}$.

Definition

The valuation v_w of formulas in a world $w = \langle A, v\rangle$ is such that

$$v_w(\Box\varphi) = \bigcap_A \{v_{w'}(\varphi)^A \mid wRw'\}$$

Truth in a world

Definition

For a filter F over L, we define the relation $\vDash_F$ for a world w:

$$w \vDash_F \varphi \text{ if, and only if, } v_w(\varphi) \in F$$

Definition

For a filter F over L, we define the relation $\vDash_F$ for a model M:

$$M \vDash_F \varphi \text{ if, and only if, } v_w(\varphi) \in F \text{ for all } w \in W_M$$

Axiom K

Theorem

Let M be a many-algebra modal model over L. Then any F-implicative world of M satisfies (under $\vDash_F$) $(\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi)$.

Theorem

Let M be a many-algebra modal model over L and F a filter. Let $w \in M$ be a F-implicative world such that A_w preserves order under $-$, and A_w has down-distribution over constants. Then w satisfies ($\vDash_F$) the axiom K, i.e. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$.

Necessitation

Theorem

Let L be a lattice and F a filter over L, then all L-models satisfy ($\models_K$) *necessitation if, and only if, F is complete-closed and sublattice-uniform in L.*

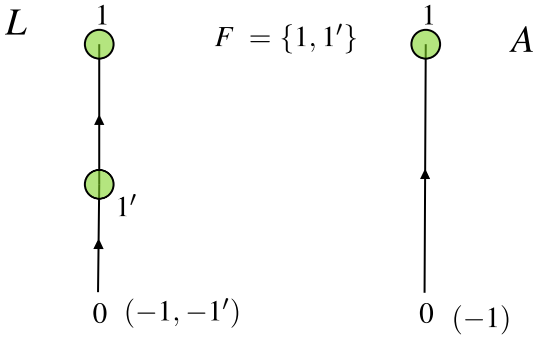
Notation

Let L be a lattice and F a filter in L, then we write L^F_{cu} *for the set of all complete sublattices X of L such that F is complete-closed in X and F is X-uniform.*

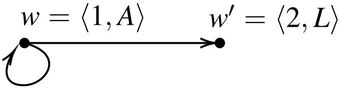
Theorem

Let L be a lattice and F a filter over L, then all L-models satisfy ($\models_F$) L^F_{cu}*-necessitation.*

Consider the lattices:

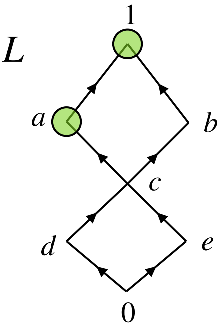


Consider $w = \langle 1, A\rangle$ and $w' = \langle 2, L\rangle$ as in the figure.
And take a valuation s such that $v_w(\alpha) = 1$ and $v_{w'}(\alpha) = 1'$ for a propositional variable α.



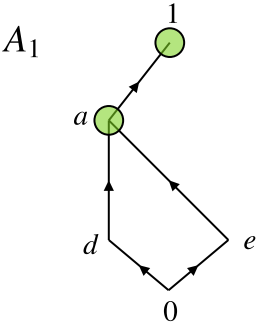
Now we observe that $w \vDash \alpha$, $w \nvDash \neg\alpha$, $w' \vDash \alpha$, $w' \nvDash \neg\alpha$.
And we have $w \nvDash \Box\alpha$, for $v_w(\Box\alpha) = \bigcap_A \{1, 1'\} = 1 \cap 0 = 0$.

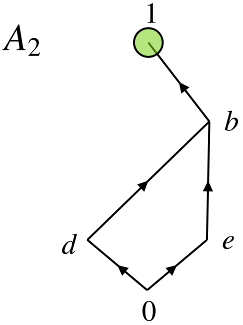
Now consider the lattice:



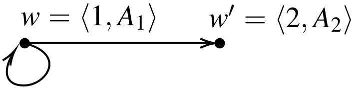
$F = \{1, a\}$

And the sublattices:





Consider the following frame:



And $v_w(\alpha) = 1$ and $v_{w'}(\alpha) = b$, for a propositional variable α. In this case we have $w \vDash \alpha$ and $w' \nvDash \alpha$. Moreover, $w \vDash \Box\alpha$, since $v_w(\Box\alpha) = a$.

Extended notion of frame

We should now differentiate **frames** and **graphs**.

1. frame: including the lattice structure and world.
2. graph: only the structure for the accessibility relation.

We can now obtain interesting frames:

1. classically increasing frame: access worlds with sublattices where $a +_{w'} b \geq a +_{w} b$ and $a ._{w'} b \leq a ._{w} b$.
2. classically decreasing frame: $a +_{w'} b \leq a +_{w} b$ and $a ._{w'} b \geq a ._{w} b$.
3. dialectic frame (mix of logic and accessibility relation): two less classical worlds w_1 and w_2 accessible to w have a common accessible w' which is more classical than w.

Further discussion

1. up/down interpretations for negation.
2. recovering traditional approaches.
3. characterization of frames.
4. general properties that imply Finite Model Property.

Thank you.

Alfredo Roque Freire
Universidade de Aveiro

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