First order doctrines as finitely bipresentable 2-categories

Axel Osmond (joint work with Ivan Di Liberti)

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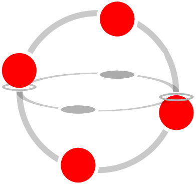

Introduction

Varieties of propositional algebras

Fragments of propositional logic correspond to varieties of propositional algebras:

- $\wedge$-**SLat** for propositional cartesian logic
- **DLat** for propositional coherent logic
- **Heyt** for propositional first order logic
- **Bool** for propositional classical logic
- also diverse varieties of residuated lattices for substructural logics: **ResLat**, **FL**, $\mathbf{FL}_0$, **BL**, **GBL**, **MV**...

Those varieties are often studied in the context of *universal algebra*.

But from the point of view of *categorical model theory*, they are also instances of *locally finitely presentable categories*.

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Filteredness and finitely accessible categories

Locally finitely presentable categories are categories having:

- small colimits (in particular, filtered colimits)
- an essentially small subcategory of finitely presented objects such that any object is a filtered colimits of finitely presented objects

(Requiring existence only of filtered colimits gets the more general notion of *finitely accessible categories*)

Filtered colimits: those indexed by category I where

$$i_1 \dashrightarrow j \dashleftarrow i_2 \qquad\qquad i_1 \rightrightarrows i_2 \dashrightarrow j$$

Finitely presented (aka compact) objects: those that are "small against filtered colimits" (whose covariant homset preserve filtered colimits).

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Locally finitely presentable enjoy a lot of pleasant properties:

- completeness and commutation of finite limits with filtered colimits,
- well (co)poweredness,
- special small object argument,
- good interactions of monomorphisms and colimits...

They also encompass actually far more examples, as:

- sets, posets,
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First order doctrines

While fragments of propositional logic correspond to varieties of propositional algebras, fragments of first order logic correspond to *doctrines*:

Those are 2-categories whose objects are *syntactic categories* associated to first order theories and functors preserving the associated internal logic.

Some instance of first order doctrines include:

- **Lex** (lex categories) - for cartesian logic (categorifying $\wedge$-**SLat**)
- **Prod** (categories with finite product) - for algebraic theories
- **Reg** (regular categories) - for regular logic
- **Coh** (coherent categories) for coherent logics (categorifying **DLat**)

Other doctrines are classes of categories defined from exactness conditions:

- **Ex** (exact categories)
- $\mathbf{Ext}_\omega$ (lextensives categories)
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Are those doctrines finitely presentable in some 2-dimensional sense ?

When categorifying a notion, several degrees of strictness are possible. A first, *enriched* version of presentability was investigated in

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However our conjectured examples required a less strict framework.

We relied rather on the recent theory of *flat pseudofunctors* developed in

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A short discussion on 2-dimensional filteredness

1-dimensional flatness

In 1-dimension, a key result is that finitely accessible categories corresponds are exactly categories of flat functors $\mathbf{Flat}[C, \mathbf{Set}]$ with $\mathcal{C}$ small.

A functor $F : \mathcal{C} \to \mathbf{Set}$ can be decomposed as conical colimit of representables, using its category of elements:

$$F \simeq \underset{(C,a)\in(\int F)^{\mathrm{op}}}{\mathrm{colim}} \; よ_C$$

(with $よ : \mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathbf{Set}]$ the Yoneda embedding).

Then flat functors can be defined equivalently as:

- those F whose left Kan extension $\mathrm{Lan}_{よ} F : [C^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set}$ is lex
- those F that are filtered colimits of representables

In particular when $\mathcal{C}$ is lex, being flat amounts to being lex.

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Decomposing pseudofunctors

In 2-dimension, this complicates a bit.

For pseudofunctors, we have a decomposition into a *weighted* bicolimit:

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However this expression is not equivalent to a conical bicolimit.

This makes impossible to detect any filteredness condition.

Is there a *conical* decomposition of pseudofunctors into representables, so we can detect eventual 2-dimensional filteredness in the indexing 2-category ?

This is the point of D.D.S. notion of σ-*bicolimits*.

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Dubuc-Descotte-Szyld theory of flatness

σ-colimits are intermediate between pseudocolimits and oplax colimits: here, only *some* transition 2-cells in the oplax cocone are invertible.

One can turn any weighted bicolimit into a conical σ-bicolimit.

D.D.S. developed a suited notion of *σ-filteredness* for σ-bicolimits.

Then they introduced a notion of *flat pseudofunctors*, equivalently:

- those whose *left biKan extension* preserves finitely weighted bilimits
- those that are σ-filtered σ-bicolimits of representable.

Hence, at first sight, a theory of 2-dimensional accessibility in this framework would rely on σ-filteredness.

However, we proved that σ-filteredness simplified into a more practical notion of *bifilteredness*.

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Bifilteredness

Definition (Kennisson)

A 2-category I is said to be *bifiltered* if:

- for any i, i' in I there exists a span $i \xrightarrow{d} i'' \xleftarrow{d'} i'$

- for a parallel pair $d, d' : i \rightrightarrows i'$, there exists $f : i' \to i''$ together with an invertible 2-cell

$$\begin{array}{ccccc} & & i' & & \\ & \overset{d}{\nearrow} & & \overset{d''}{\searrow} & \\ i & & \overset{\alpha}{\simeq} & & i'' \\ & \underset{d'}{\searrow} & & \underset{d''}{\nearrow} & \\ & & i' & & \end{array}$$

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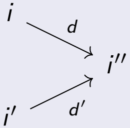
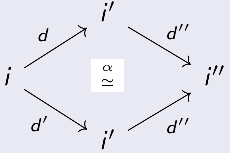
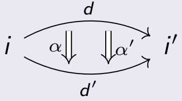
- for any i, i' in I there exists a span 

- for a parallel pair $d, d' : i \rightrightarrows i'$, there exists $f : i' \to i''$ together with an invertible 2-cell 

- for a pair of parallel 2-cells 

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Bifiltered reformulation of D.D.S.

Using a suited 2-dimensional form of *cofinality*, we observed the following:

Lemma (DL.O. 1.6.8)

Any σ-filtered σ-bicolimit is equivalent to a conical bifiltered bicolimit.

D.D.S. characterization of flat pseudofunctors could then be simplified:

Theorem (DL.O. 3.1.6)

Let $\mathcal{C}$ be a small 2-category. Then for a pseudofunctor $F : \mathcal{C} \to \mathbf{Cat}$ we have the following equivalences

- *F is flat, that is,* $\mathrm{biLan}_{よ} F$ *is bilex*
- *F decomposes as a bifiltered bicolimit of representables.*

Knowing this, it appear we can ground a theory of 2-dimensional accessibility on D.D.S. results but involving only bifiltered bicolimits.

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Bi-accessible and bipresentable 2-categories

Bicompact objects

First, what should be the analogs of finitely presented objects ?

Definition

An object K in a 2-category $\mathcal{B}$ is *bicompact* if for any bifiltered 2-category I and any 2-functor $F : I \rightarrow \mathcal{B}$, we have an equivalence of categories

$$\mathcal{B}[K, \underset{I}{\text{bicolim}}\, F] \simeq \underset{i \in I}{\text{bicolim}}\, \mathcal{B}[K, F(i)]$$

(In fact they enjoy the same property against σ-filtered σ-colimits.)

Proposition

Any finitely weighted bicolimit of bicompacts is bicompact.

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$$\mathcal{B}[K, \underset{I}{\mathrm{bicolim}}\, F] \simeq \underset{i \in I}{\mathrm{bicolim}}\, \mathcal{B}[K, F(i)]$$

(In fact they enjoy the same property against σ-filtered σ-colimits.)

Proposition

Any finitely weighted bicolimit of bicompacts is bicompact.

Bicompact objects

First, what should be the analogs of finitely presented objects ?

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Finitely bi-accessible 2-categories

Definition

A 2-category $\mathcal{B}$ will be said *finitely bi-accessible* if

- $\mathcal{B}$ has bifiltered bicolimits,
- there is an essentially small (1,2)-full sub-2-category $\mathcal{B}_0 \hookrightarrow \mathcal{B}$ consisting of bicompact objects such that for any B in $\mathcal{B}$ is a bifiltered bicolimit of objects in $\mathcal{B}_0$.

In fact, one can take the full sub-2-category of all bicompact objects.

Definition

A 2-category is said to be *finitely bipresentable* if it is finitely bi-accessible and has all small weighted bicolimits.

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A 2-category is said to be *finitely bipresentable* if it is finitely bi-accessible and has all small weighted bicolimits.

The canonical pseudococone

For B in $\mathcal{B}$ finitely bi-accessible, one can consider the *canonical pseudocone* of B given by the pseudoslice $\mathcal{B}_\omega \downarrow B$.

Its objects are pairs (K, a) with $a : K \to B$, and a morphism $(K_1, a_1) \to (K_2, a_2)$ is a pair (k, ϕ) coding for an invertible 2-cell

$$\begin{array}{ccc} K_1 & \xrightarrow{k} & K_2 \\ & \overset{\phi}{\simeq} & \\ {\scriptstyle a_1} \searrow & & \swarrow {\scriptstyle a_2} \\ & B & \end{array}$$

Its 2-cells are $\alpha : k_1 \Rightarrow k_2$ such that $\phi_2 a_2 * \alpha = \phi_1$.

Proposition

If $\mathcal{B}$ is finitely bi-accessible, then for any B the canonical pseudocone $\mathcal{B}_\omega \downarrow B$ is bifiltered and

$$B \simeq \text{bicolim}\ \mathcal{B}_\omega \downarrow B$$

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The binerve embedding

The formula above says that the inclusion $\iota_\omega : \mathcal{B}_\omega \hookrightarrow \mathcal{B}$ is *bidense*.

Equivalently, we have a 2-embedding into the pseudofunctors 2-category

$$\mathcal{B} \xhookrightarrow{\nu} \mathbf{ps}[(\mathcal{B}_\omega)^{\mathrm{op}}, \mathbf{Cat}]$$

sending B to $\mathcal{B}[\iota_\omega, B]$, the restriction of the representable at B along ι_ω.

Moreover, for $\mathcal{B}_\omega \downarrow B$ is bifiltered, $\mathcal{B}[\iota_\omega, B]$ is flat. Hence :

Proposition (DL.O. 3.1.9)

For any finitely accessible category $\mathcal{B}$, ν reduces to a biequivalence

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Characterization with bistrong generator

For finitely bipresentable 2-category, it can be sufficient to exhibit a weaker kind of generator containing just "enough" bicompacts objects.

Definition

A a small sub 2-category $\iota : \mathcal{G} \hookrightarrow \mathcal{B}$ is a *strong generator* if its binerve

$$\mathcal{B} \xrightarrow{\nu} \mathbf{ps}[\mathcal{G}^{\mathrm{op}}, \mathbf{Cat}]$$

is biconservative, that is, reflects equivalences.

Theorem (DL.O. 2.4.3)

Let $\mathcal{B}$ be a 2-category with weigthed bicolimits. Then the following are equivalent:

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For a finitely accessible $\mathcal{B}$, the embedding $\nu_{\mathcal{B}}$ restricts to an equivalence:

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When $\mathcal{B}$ is finitely bipresentable, $(\mathcal{B}_\omega)^{\mathsf{op}}$ is *bilex*, whence:

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2-dimensional Gabriel-Ulmer duality

Definition

The tricategory **biLex** has objects small 2-categories with weighed finite bilimits. 1-cells are pseudofunctors preserving finite bilimits, 2-cells are pseudonatural transformations and 3-cells are modifications.

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The tricategory $\mathbf{biP}_\omega$ has objects finitely bipresentable 2-categories. 1-cells are right biadjoints preserving bifiltered bicolimits, 2-cells are pseudonatural transformations and 3-cells are modifications.

Theorem (DL.O. 4.3.3)

There is a three-equivalence of tricategories

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Examples

Cat is finitely bipresentable

Lemma

In **Cat***, finite categories are bicompact.*

Not all bicompact objects in **Cat** are finite. For example, the monoid $\mathbb{N}$ - seen as a 1-object category - is the coinserter of the diagram below and thus is bicompact:

$$\{*\} \underset{\mathsf{id}}{\overset{\mathsf{id}}{\rightrightarrows}} \{*\} \longrightarrow \mathbb{N}$$

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Bifinitary pseudomonads

A second class of examples will come from 2-categories of *pseudoalgebras*.

Pseudomonads are 2-functors equipped with pseudonatural unit and multiplication satisfying monad identities up to canonical invertible 2-cells.

Similarly, replacing strict equalities in the definition of algebras and morphisms of algebras by invertible 2-cells gives the notion of pseudoalgebras and pseudomorphisms.

A pseudomonad is said to be *bifinitary* if it preserves bifiltered bicolimits.

The following generalizes Blackwell, Kelly and Power result for 2-categories *strict* algebras and pseudomorphisms for *strict* 2-monad:

Theorem (O.)

*Let T be a bifinitary pseudomonad on a bicocomplete 2-category $\mathcal{C}$. Then T-**psAlg** is bicocomplete. Moreover the forgetful functor $U_T : T\text{-}\mathbf{psAlg} \to \mathcal{C}$ creates bicolimits that T preserves.*

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Bipresentability of 2-categories of pseudo-algebras

As well as algebras for a finitary monad over a finitely presentable category are finitely presentable, we have the following:

Theorem (DL.O. 5.2.2)

*Let $\mathcal{B}$ be a finitely bipresentable 2-category and T a bifinitary pseudomonad on $\mathcal{B}$. Then T-**psAlg** is also finitely bipresentable. Moreover $U_T : T\text{-}\mathbf{psAlg} \to \mathcal{B}$ preserves bifiltered bicolimits.*

Sketch of the proof :

- T-**psAlg** has bifiltered bicolimits preserved by U_T for T is bifinitary.

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The case of Lex

Lex is the 2-category of small categories with finite limits.
It is pseudomonadic on **Cat** through the free lex completion.

Lemma

Lex *is closed in* **Cat** *under bifiltered bicolimits.*

This uses the description of pseudocolimits in **Cat** as localization of oplax-colimit; the argument is that the finiteness of the diagram interacts well with the filteredness condition.

Theorem (DL.O. 5.3.3)

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Further exactness conditions

What about other doctrines as **Reg**, **Coh**, **Ex**... ?

And other categories defined through exactness conditions, **Ext**, **Pretop** ?

Those examples can be captured at once through the theory of *lex colimits*.

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Exactness conditions through lex-colimits

In the following, Φ denotes a class of finite weights $W : I^{op} \to \mathbf{Set}$.

For Φ and a category $\mathcal{C}$, consider the full subcategory $\Phi_I(\mathcal{C}) \hookrightarrow \widehat{\mathcal{C}}$ consisting of all Φ-weighted colimits of representables.

Definition (Garner, Lack)

A small $\mathcal{C}$ is Φ-*lex-cocomplete* if it is lex and has all Φ-weighted colimits.

This amounts to requiring the existence of a left adjoint

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A small $\mathcal{C}$ is Φ-*lex-cocomplete* if it is lex and has all Φ-weighted colimits.

This amounts to requiring the existence of a left adjoint

$$\mathcal{C} \underset{\iota_{\mathcal{C}}}{\overset{L_{\mathcal{C}}}{\rightleftarrows}} \Phi_I(\mathcal{C}) \qquad (L_{\mathcal{C}} \dashv \iota_{\mathcal{C}})$$

A Φ-lex-cocomplete category is Φ-*exact* if this left adjoint is lex.

This amounts to saying that $(\mathcal{C}, L_{\mathcal{C}})$ bears a structure of pseudo-algebra for the pseudomonad Φ_I on $\mathbf{Lex}$.

Any Φ-Ex is finitely bipresentable

Theorem (DL.O. 5.4.5)

For Φ a class of finite weights, the 2-category of Φ-exact categories and Φ-exact functors is finitely bipresentable.

- The 2-category Φ-**Ex** is the 2-category of pseudo-algebras Φ_I-**psAlg**.
- It suffices to prove Φ_I to be finitary, which amounts to proving the forgetful functor $U_\Phi : \Phi_I\text{-}\mathbf{psAlg} \to \mathbf{Lex}$ to be finitary.
- Using that all weights in Φ are finite, we show that U_Φ preserves bifiltered bicolimits of free Φ_I pseudo-algebras

$$U_\Phi \Phi_I(\underset{i \in I}{\mathrm{bicolim}}\, F(i)) \simeq \underset{i \in I}{\mathrm{bicolim}}\, U_\Phi \Phi_I F(i)$$

- For a bifiltered $F : I \to \Phi_I\text{-}\mathbf{psAlg}$ the adjunctions $L_{F(i)} \dashv \iota_{F(i)}$ induce an adjunction in **Lex** between the bifiltered bicolimits

$$\underset{i \in I}{\mathrm{bicolim}}\, F(i) \;\underset{\underset{i\in I}{\mathrm{bicolim}}\, \iota_{F(i)}}{\overset{\underset{i\in I}{\mathrm{bicolim}}\, L_{F(i)}}{\rightleftarrows}}\; U_\Phi \Phi_I(\underset{i \in I}{\mathrm{bicolim}}\, F(i)) \qquad \perp$$

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with $\operatorname*{bicolim}_{i \in I} L_{F(i)}$ and $\operatorname*{bicolim}_{i \in I} \iota_{F(i)}$.

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Examples amongst Φ-exact categories

Corollary

The following 2-categories are finitely bipresentable:

- **Reg**: *small regular categories and regular functors;*
- **Ex**: *small (Barr)-exact categories and exact functors;*
- **Coh**: *small coherent categories and coherent functors;*
- $\mathbf{Ext}_\omega$: *small finitely-extensive categories and functors preserving finite coproducts;*
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Thank you !

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