From residuated lattices to ℓ-groups via free nuclear preimages

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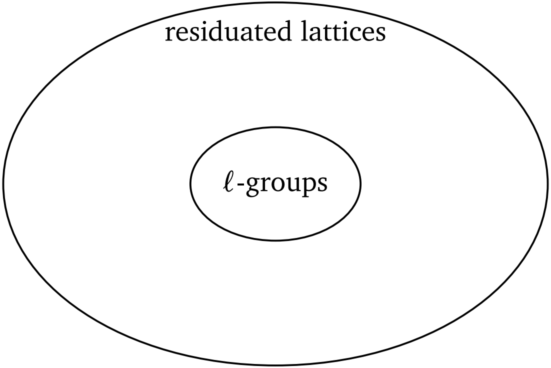

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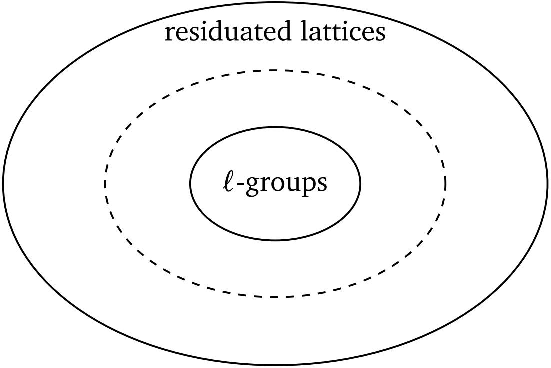

Which pomonoids arise from cancellative pomonoids?

Which sℓ-monoids arise from cancellative sℓ-monoids?

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We give a full answer.

Which $s\ell$-monoids arise from cancellative $s\ell$-monoids?

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We give a partial answer.

A **partially ordered monoid (pomonoid)** is a partially ordered algebra $\mathbf{M} = \langle M, \leq, \cdot, \mathsf{e} \rangle$ such that $\langle M, \cdot, \mathsf{e} \rangle$ is a monoid and multiplication is isotone.

A **semilattice-ordered monoid (sℓ-monoid)** is an algebra $\mathbf{M} = \langle M, \vee, \cdot, \mathsf{e} \rangle$ such that $\langle M, \cdot, \mathsf{e} \rangle$ is a monoid, $\langle M, \vee \rangle$ is a join semilattice, and

$$a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot b), \qquad (a \vee b) \cdot c = (a \cdot c) \vee (b \cdot c).$$

A **residuated pomonoid** moreover has two division operations such that

$$b \leq a \backslash c \iff a \cdot b \leq c \iff a \leq c / b.$$

A **residuated lattice** is both a lattice and a residuated pomonoid.

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1. Consider the negative cone $\mathbf{G}^-$. This is an integral residuated lattice:

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2. Consider some u in $\mathbf{G}^-$. Then the interval $[u, \mathsf{e}] \subseteq \mathbf{G}^-$ is an MV-algebra:

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Theorem (Dvurečenskij). Every pseudo MV-algebra arises from some ℓ-group in the same way. [These are "non-commutative MV-algebras".]

The negative cone construction is an example of a **conuclear image**.

A **conucleus** on a pomonoid $\mathbf{M}$ is an interior operator σ such that

$$\sigma(a) \cdot \sigma(b) \leq \sigma(a \cdot b), \qquad \sigma(\mathsf{e}) = \mathsf{e}.$$

The σ-open elements of $\mathbf{M}$ form a subpomonoid $\mathbf{M}_\sigma = \langle M_\sigma, \leq, \cdot, \mathsf{e} \rangle$.

If σ is a conucleus on a residuated lattice $\mathbf{L}$, then $\mathbf{L}_\sigma$ is a residuated lattice which is subalgebra of $\mathbf{L}$ w.r.t. $\vee, \cdot, \mathsf{e}$.

The unit interval construction is an example of a **nuclear image**.

A **nucleus** on a pomonoid $\mathbf{M}$ is a closure operator such that

$$\gamma(a) \cdot \gamma(b) \leq \gamma(a \cdot b).$$

The γ-closed elements of $\mathbf{M}$ form a pomonoid $\mathbf{M}_\gamma = \langle M_\gamma, \leq, \cdot_\gamma, \gamma(\mathsf{e}) \rangle$ with

$$a \cdot_\gamma b := \gamma(a \cdot b).$$

If γ is a nucleus on a residuated lattice $\mathbf{L}$, then $\mathbf{L}_\gamma$ is a residuated lattice which is a subalgebra of $\mathbf{L}$ w.r.t. $\wedge, \backslash, /$.

Theorem (Galatos & Tsinakis). Every (commutative) GMV-algebra arises as the nuclear image of a kernel image of an (Abelian) ℓ-group.

GMV-algebras form a variety of residuated lattices which generalizes MV-algebras by dropping integrality, commutativity, and boundedness.

Here a **kernel** is a conucleus whose image is downward closed.

Which algebras arise as nuclear images of conuclear images of ℓ-groups?

$$\ell\text{-group} \xrightarrow{\text{conucleus } \sigma} \; ? \; \xrightarrow{\text{nucleus } \gamma} \; ?$$

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$$\ell\text{-group} \xrightarrow{\text{conucleus } \sigma} \; ? \; \xrightarrow{\text{nucleus } \gamma} \; ?$$

By-product: which quasivarieties are closed under nuclear images?

A pomonoid is (order) **cancellative** if

$$z \cdot x \leq z \cdot y \implies x \leq y, \qquad x \cdot z \leq y \cdot z \implies x \leq y.$$

A pomonoid is **integrally closed** if

$$y \cdot x \leq y \implies x \leq \mathsf{e}, \qquad x \cdot y \leq y \implies x \leq \mathsf{e}.$$

Cancellative $\implies$ integrally closed. Integral $\implies$ integrally closed.

Finite integrally closed $\iff$ finite integral.

Fact. Conuclear images preserve cancellativity. Nuclear images preserve the property of being integrally closed. Therefore:

$$\text{pogroup} \xrightarrow{\text{conucleus } \sigma} \text{cancellative} \xrightarrow{\text{nucleus } \gamma} \text{integrally closed}$$

Each commutative cancellative pomonoid (sℓ-monoid) $\mathbf{M}$ embeds into an Abelian **pogroup (ℓ-group) of fractions G** where

$$\text{each } x \in \mathbf{G} \text{ has the form } x = a^{-1}b \text{ for some } a, b \in \mathbf{M}.$$

If $\mathbf{M}$ is residuated, then there is a conucleus σ on $\mathbf{G}$ such that $\mathbf{M} \cong \mathbf{G}_\sigma$:

$$\sigma(a^{-1}b) := a \backslash_{\mathbf{M}} b.$$

Theorem (Montagna & Tsinakis). Commutative cancellative RLs (RPs) are precisely the conuclear images of Abelian ℓ-groups (pogroups).

Beyond the commutative case, things are more complicated. It is difficult to describe even which cancellative monoids embed into a group.

Core problem: given an integrally closed pomonoid $\mathbf{M} = \langle M, \leq, \cdot, \mathsf{e} \rangle$, show that it is the nuclear image of a cancellative pomonoid.

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Let $\langle M^*, \leq, \circ, \varepsilon \rangle$ be the free monoid (monoid of words) over an arbitrary pomonoid $\mathbf{M}$. Words will be written as

$$[a_1, \ldots, a_n] = [a_1] \circ [a_2] \circ \ldots \circ [a_n].$$

M^* comes with a **multiplication map** $\gamma : M^* \to M$:

$$\gamma([a_1, \ldots, a_n]) := a_1 \cdot \ldots \cdot a_n, \qquad \gamma(\varepsilon) := \mathsf{e}.$$

This yields a map $[\gamma] : M^* \to M^*$, namely $[\gamma](w) := [\gamma(w)]$.

Define the following preorder on M^*:

$$
\begin{aligned}
u \sqsubseteq \varepsilon &\iff u = \varepsilon, \\
u \sqsubseteq [a] &\iff \gamma(u) \leq a \text{ in } \mathbf{M}, \\
u \sqsubseteq [a_1, \ldots, a_n] &\iff u_1 \sqsubseteq [a_1] \text{ and } \ldots \text{ and } u_n \sqsubseteq [a_n] \\
&\qquad \text{for some decomposition } u_1 \circ \ldots \circ u_n = u.
\end{aligned}
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Note that some of the u_i's in the decomposition might be empty.

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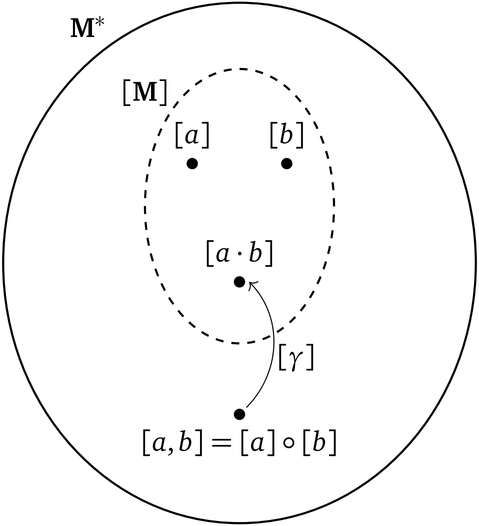
Quotienting the preordered monoid $\langle M^*, \sqsubseteq, \circ, \varepsilon \rangle$ to a partially ordered structure yields the pomonoid $\mathbf{M}^*$. This is **never** cancellative:

$$\varepsilon \circ u = [\mathsf{e}] \circ u \text{ unless } u = \varepsilon.$$

Collapsing ε and $[\mathsf{e}]$ yields the pomonoid $\mathbf{M}^+$.


$\mathbf{M}^*$
$[\mathbf{M}]$
$[a]$
$[b]$
$[a \cdot b]$
$[\gamma]$
$[a,b] = [a] \circ [b]$

A **(unital) nuclear pomonoid** or **sℓ-monoid** is pomonoid or sℓ-monoid $\mathbf{M}$ equipped with a (unital) nucleus γ. Here **unital** means that $\gamma(\mathsf{e}) = \mathsf{e}$.

The **(unital) nuclear image functor** from the category of (unital) nuclear pomonoids to the category of pomonoids:

$$\langle \mathbf{M}, \gamma \rangle \mapsto \mathbf{M}_\gamma$$

The **free (unital) nuclear preimage** functor is its left adjoint.

Fact. $\mathbf{M}^*$ ($\mathbf{M}^+$) is the free (unital) nuclear preimage of $\mathbf{M}$. The unit of the adjunction is the map $a \mapsto [a]$. This is an isomorphism: $(\mathbf{M}^*)_{[\gamma]} \cong \mathbf{M}$.

Fact. Each equivalence class of $\mathbf{M}^*$ has a unique shortest element. (If $\mathbf{M}$ is integral: remove subwords of the form $[\mathsf{e}]$ unless the whole word is $[\mathsf{e}]$.)

Mutatis mutandis, the same construction works in the commutative case.

Theorem.

Nuclear images of cancellative pomonoids
$=$
integrally closed pomonoids.

Theorem.

Nuclear images of (commutative) cancellative pomonoids

=

(commutative) integrally closed pomonoids.

Theorem.

Nuclear images of (commutative) [integral] cancellative pomonoids

=

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Nuclear images of (commutative) [integral] cancellative pomonoids
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(commutative) integrally closed [integral] pomonoids.

Proof. $\mathbf{M}^+$ is cancellative if $\mathbf{M}$ is an integrally closed pomonoid:

It suffices to show that $[a] \circ u \sqsubseteq [a] \circ v$ in $\mathbf{M}^+$ implies $u \sqsubseteq v$.

If $[a] \circ u_1 \sqsubseteq [a]$ and $u_2 \sqsubseteq v$ for some $u_1 \circ u_2 = u$, then $a \cdot \gamma(u_1) \leq a$, so $\gamma(u_1) \leq \mathsf{e}$ because $\mathbf{M}$ is integrally closed. Thus $u = u_1 \circ u_2 \sqsubseteq [\mathsf{e}] \circ v \sqsubseteq v$.

On the other hand, if $\varepsilon \sqsubseteq [a]$ and $[a] \circ u \sqsubseteq v$, then $u = \varepsilon \circ u \sqsubseteq [a] \circ u \sqsubseteq v$.

Theorem.

Nuclear images of (commutative) [integral] cancellative pomonoids
$$=$$
(commutative) integrally closed [integral] pomonoids.

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Core problem, sℓ-version: given an integrally closed sℓ-monoid **M**, show that it is the nuclear image of a cancellative sℓ-monoid.

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Define $\mathrm{Id}_\omega \mathbf{M}^*$ as the nuclear sℓ-monoid of non-empty finitely generated downsets of $\mathbf{M}^*$ with multiplication

$$X * Y := {\downarrow}(X \cdot Y)$$

and with the nucleus

$$[\gamma]({\downarrow}\{w_1, \dots, w_n\}) := [\gamma(w_1) \vee \cdots \vee \gamma(w_n)].$$

Fact. $\mathrm{Id}_\omega \mathbf{M}^*$ ($\mathrm{Id}_\omega \mathbf{M}^+$) is the free (unital) nuclear sℓ-preimage of $\mathbf{M}$.

Fact. $\mathrm{Id}_\omega \mathbf{M}^*$ is a residuated lattice if $\mathbf{M}$ is a finite residuated lattice.

Theorem.

Nuclear images of integral cancellative $s\ell$-monoids
=
integral $s\ell$-monoids.

Theorem.

Nuclear images of commutative (integral) cancellative $s\ell$-monoids
=
commutative integrally closed (integral) $s\ell$-monoids
which satisfy the square condition.

Theorem.

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$=$
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Theorem.

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$=$
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which satisfy the square condition.

Conjecture.

Nuclear images of cancellative $s\ell$-monoids
$=$
integrally closed $s\ell$-monoids?

Theorem.

Finite nuclear images of integral cancellative RLs
=
finite integral RLs.

Theorem.

Finite nuclear images of conuclear images of Abelian ℓ-groups
=
finite nuclear images of commutative integral cancellative RLs
=
finite integral CRLs with the square condition.

In fact, something stronger holds.

Theorem.

Finite nuclear images of distributive integral cancellative RLs with
$x(y \wedge z) = xy \wedge xz$ and $(x \wedge y)z = xz \wedge yz$
$=$
finite integral RLs.

Conjecture.

Finite nuclear images of integral conuclear images of ℓ-groups
$=$
finite nuclear images of integral RLs
$=$
finite nuclear images of integral conuclear images of ℓ-groups w.r.t. a conucleus σ such that $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$?

Which quasivarieties are preserved under nuclear images?

A **quasivariety of pomonoids (sℓ-monoids)** is a class of pomonoids axiomatized by **quasi-inequations**, i.e. implications of the form

$$t_1 \leq u_1 \;\&\; \dots \;\&\; t_n \leq u_n \implies t \leq u,$$

where the t's and u's are monoidal (sℓ-monoidal) terms.

Fact. A class of pomonoids (sℓ-monoids) is a quasivariety if and only if it is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_{\mathrm{U}}$.

Example. Cancellative pomonoids form a quasivariety:

$$x \cdot y \leq x \cdot z \implies y \leq z, \qquad x \cdot z \leq y \cdot z \implies x \leq y.$$

Example. Integrally closed pomonoids form a quasivariety:

$$x \cdot y \leq x \implies y \leq \mathsf{e}, \qquad x \cdot y \leq y \implies x \leq \mathsf{e}.$$

Let us call a quasi-inequality **simple** if it has the form

$$t_1 \leq x_1 \;\&\; \dots \;\&\; t_n \leq x_n \implies t \leq u,$$

where $x_1, \dots, x_n$ are variables (not necessarily distinct).

Example. Being integrally closed is a simple condition.

Non-example. Being cancellative is not a simple condition.

Theorem. A quasivariety of pomonoids (sℓ-monoids) is closed under nuclear images if and only if it is axiomatized by simple quasi-inequalities.

Equivalently, a class of pomonoids (sℓ-monoids) is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_{\mathrm{U}}$, and $\mathbb{N}$ if and only if it is axiomatized by simple quasi-inequalities.

Theorem. Let K be a quasivariety of pomonoids (sℓ-monoids). Then the class $\mathbb{N}(\mathsf{K})$ is axiomatized by the simple quasi-inequalities valid in K.

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Theorem. Let K be a quasivariety of pomonoids (sℓ-monoids). Then the class $\mathbb{N}(\mathsf{K})$ is axiomatized by the simple quasi-inequalities valid in K.

Proof. This reflects the product distributivity of the free nuclear preimage:

$$u \sqsubseteq v_1 \circ v_2 \implies u_1 \sqsubseteq v_1 \text{ and } u_2 \sqsubseteq v_2 \text{ for some } u_1 \circ u_2 = u,$$

and the distributivity of the free semilattice-ordered nuclear preimage:

$$u \sqsubseteq v_1 \vee v_2 \implies u_1 \sqsubseteq v_1 \text{ and } u_2 \sqsubseteq v_2 \text{ for some } u_1 \vee u_2 = u.$$

Back to our original question: relating pomonoids and pogroups.

Theorem.

Nuclear images of [integral] subpomonoids of pogroups

$=$

integrally closed [integral] pomonoids.

Proof strategy: proof-theoretic, through a normalization procedure.

Conjecture.

Nuclear images of [integral] sub-sℓ-monoids of ℓ-groups

$=$

integrally closed [integral] sℓ-monoids?

Thank you for your attention!