Generalized canonical extensions for frames

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22nd of June 2022

[1] Joint work with Tomáš Jakl

The main thesis

In this talk I will make the following point.

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The canonical extension of a frame is characterized via a universal-like algebraic property.

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The canonical extension of a frame is characterized via a universal-like algebraic property. Several structures important in pointfree topology enjoy natural generalizations of that property.

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$$\mathrm{pf} : \mathbf{Distr}^{op} \to \mathbf{Spec},$$

which sends each lattice to the set of its prime filers, topologized by setting the opens to be $\varphi_D(a) = \{P \in \mathrm{pf}(D) : a \in P\}$.

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maps each spectral space to the lattice of its compact opens.

Fact

This dual equivalence holds if and only if the Prime Ideal Theorem holds.

Dualities review

We have a dual adjunction between the categories of *frames* (complete lattices where arbitrary joins distribute over finite meets) and topological spaces. One adjoint is

$$\Omega : \mathbf{Top} \to \mathbf{Frm}^{op},$$

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$$\Omega : \mathbf{Top} \to \mathbf{Frm}^{op},$$

which maps a space to the ordered collection of its opens. Its right adjoint is

$$\mathrm{pt} : \mathbf{Frm}^{op} \to \mathbf{Top},$$

which maps a frame to the set of its completely prime filters, topologized by setting the opens to be $\varphi_L(a) = \{P \in \mathrm{pt}(L) : a \in P\}$. A filter is *completely prime* when it is inaccessible by arbitrary joins.

Fact

This adjunction maximally restricts to a dual equivalence between *spatial* frames and sober spaces.

Saturated sets

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Fact

For a space X we have $\mathrm{Sat}(X) = \mathcal{U}(X)$.

Canonical extensions of distributive lattices

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The study of canonical extensions goes back to [6], where the concept was introduced for Boolean algebras. For a distributive lattice D we have an embedding

$$\begin{aligned} D &\hookrightarrow \mathcal{U}(\mathrm{pf}(D)) \\ a &\mapsto \varphi_D(a) \end{aligned}$$

The motivation behind the concept of canonical extension is to describe this embedding purely algebraically and in a choice-free manner. See [4, 3], for more on canonical extensions of distributive lattices. See [1] for a broad view on the notion of canonical extension.

Canonical extensions of distributive lattices

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Proposition

The canonical extension $e : D \to D^{\delta}$ of a distributive lattice exists and it is unique.

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We have a complete lattice surjection $D^\delta \to \mathcal{U}(\mathrm{pf}(D))$ acting on generators as $e(d) \mapsto \varphi(d)$. The Prime Ideal Theorem is equivalent to this being an isomorphism.

Canonical extensions of frames

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For a frame L, we give a definition of the canonical extension of a frame equivalent to that is [5]. A *canonical extension* of L is a map $e : L \to L^\delta$ into a complete lattice L^δ such that the following hold.

- *Meet-generativity*: The collection $\{e[a] : a \in L\}$ meet-generates L^δ.

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- SO-*Density*: The collection $\{\bigwedge e[F] : F \text{ is Scott-open}\}$ join-generates L^δ.

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- SO-*Canonicity*: When $F \subseteq L$ is a Scott-open filter we have that $\bigwedge e[F] \subseteq e(a)$ implies that $a \in F$.

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Canonical extensions of frames

The following was proven in [5]. Here $\mathrm{SO}(L)$ denotes the ordered collection of Scott-open filters of L. Here $\mathfrak{I}$ denotes closure under arbitrary intersections.

Theorem

For a frame L a canonical extension exists and it is unique. This is isomorphic the following.

$$L \to \mathfrak{I}(\mathrm{SO}(L))^{op}$$
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For frames the extension map is not necessarily injective. See [5] for more on this issue.

Generalizing canonical extensions for frames

We have the following useful result.

Theorem

For any collection $\mathcal{F}$ of filters of a frame L we have that the map

$$e_{\mathcal{F}} : L \to \mathfrak{I}(\mathcal{F})^{op}$$
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is unique with the properties of meet generativity, $\mathcal{F}$-density, and $\mathcal{F}$-canonicity.

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Generalizing canonical extensions for frames

When $\mathcal{F}$ is the collection $\mathrm{CP}(L)$ of completely prime filters this gives the following result.

Proposition

For a frame L we have that the map

$$e_{\mathrm{CP}} : L \to \mathcal{U}(\mathrm{pt}(L))$$
$$A \mapsto \varphi_L(a)$$

is unique with meet-generativity, CP-density, and CP-canonicity.

A remark on choice

We have the following fact.

Proposition

The structure $\mathfrak{I}(\mathsf{SO}(L))$ is a frame. There is a frame surjection $s : \mathfrak{I}(\mathsf{SO}(L)) \to \mathcal{U}(\mathsf{pt}(L))$ defined on generators as $e(a) \mapsto \varphi_L(a)$. The Prime Ideal Theorem is equivalent to s being an isomorphism.

Fitted sublocales

We can obtain some other results by varying what $\mathcal{F}$ is in our general construction. For a frame L a meet $\bigwedge_i x_i$ is called *strongly exact* if it is preserved by all frame morphisms.

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Theorem

For a frame L the collection $\mathsf{SE}(L)$ of strongly exact filters is a frame. It is also anti-isomorphic to the frame $\mathsf{S}_{fit}(L)$ of fitted sublocales of L.

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Fitted sublocales are the pointfree version of saturated subsets.

Fitted sublocales

This gives us an algebraic description of the coframe of fitted sublocales of a frame.

Proposition

For a frame L the embedding

$$L \to \mathsf{S}_{fit}(L)$$
$$a \mapsto \mathfrak{o}(a),$$

assigning to each element its open sublocale, is the unique one satisfying meet-generativity, SE-density, and SE-canonicity.

Joins of closed sublocales

Let us set $\mathcal{F}$ as the collection of exact filters. A meet $\bigwedge_i x_i$ in a frame is *exact* if for every y we have $(\bigwedge_i x_i) \vee y = \bigwedge_i (x_i \vee y)$. A filter is *exact* if it is closed under all exact meets. We have the following ([7]).

Theorem

For a frame L the collection $\mathrm{E}(L)$ of exact filters is a frame. Furthermore, it is isomorphic to the frame $\mathrm{S}_c(L)$ of joins of closed sublocales.

Joins of closed sublocales

We have then found an algebraic description of the frame of joins of closed sublocales.

Proposition

For a frame L the embedding

$$L \to \mathsf{S}_c(L)^{op}$$
$$a \mapsto \mathfrak{c}(a),$$

assigning to each element its closed sublocale, is the unique one with meet generativity, E-density, and E-canonicity.

The Booleanization

For a frame L we say that a filter F is *Boolean* if it is of the form $\{x \in L : \neg x \leq \neg\neg y\}$ for some $y \in L$. We call $\mathrm{B}(L)$ the collection of Boolean filters of L.

The Booleanization

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Closed filters

A filter of a frame is *closed* if it is of the form $\{x \in L : x \vee a = 1\}$. These are precisely the kernel filters of closed sublocales.

Fact

There is an order-reversing bijection between the closed filters of L and its closed sublocales.

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Since fitted sublocales are closed under intersections, they form a closure system. We call fitt the closure operator associated with them.

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There is an order-reversing bijection between intersections of closed filters and *fittings* of joins of closed sublocales. Furthermore, $\mathrm{fitt}[\mathsf{S}_c(L)]$ is the Booleanization of $\mathsf{S}_{fit}(L)$.

The Booleanization of the frame of fitted sublocales

Proposition

For a frame L the embedding

$$\begin{aligned} L &\to \mathrm{fit}[\mathsf{S}_c(L)^{op}] = \mathfrak{B}(\mathsf{S}_{fit}(L)) \\ a &\mapsto \mathrm{fitt}(\mathfrak{c}(a)), \end{aligned}$$

assigning to each element its closed sublocale, is the unique one with meet generativity, C-density, and C-canonicity.

Summary so far

So far, we have seen that the following structures all enjoy variations of the universal-like algebraic characterization of the canonical extension.

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- $\mathfrak{B}(L)$, the Booleanization of L.
- $\mathfrak{B}(\mathsf{S}_{fit}(L))$, the Booleanization of the fitted sublocales of L.

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- $\mathfrak{B}(L)$, the Booleanization of L.
- $\mathfrak{B}(\mathsf{S}_{fit}(L))$, the Booleanization of the fitted sublocales of L.

Now, our last step will be to look at the connection between these.

Canonical extensions and sublocales

Finally we wish to highlight another connection between locale theory and the theory of canonical extensions.

Proposition

Suppose that $\mathcal{F}$ is a collection of filters closed under the operation

$$F \mapsto \{x \vee f : f \in F\}.$$

Then, there is a sublocale inclusion $\mathfrak{I}(\mathcal{F}) \subseteq \mathsf{Filt}(L)$.

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Then, there is a sublocale inclusion $\mathfrak{I}(\mathcal{F}) \subseteq \mathrm{Filt}(L)$.

All the collections of filters that we have seen up to now satisfy this condition. Therefore we have several sublocale inclusions among our variations of the canonical extension of a frame.

A poset of sublocale inclusions between extensions

In fact, we have got the following diagram of sublocale inclusions.

$$\mathfrak{B}(\mathsf{S}_{fit}(L)) \xrightarrow{\subseteq} \mathsf{S}_c(L) \xrightarrow{\subseteq} \mathsf{S}_{fit}(L)$$

$$\mathfrak{B}(L) \xrightarrow{\subseteq} \mathsf{S}_{fit}(L) \xrightarrow{\subseteq} \mathsf{Filt}(L)$$

$$\mathcal{U}(\mathsf{pt}(L)) \xrightarrow{\subseteq} L^{\delta} \xrightarrow{\subseteq} \mathsf{S}_{fit}(L)$$

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 & & & \searrow^{\subseteq} & & & \\
 & & \mathfrak{B}(L) & \xrightarrow{\subseteq} & \mathsf{S}_{fit}(L) & \xrightarrow{\subseteq} & \mathrm{Filt}(L) \\
 & & & \nearrow^{\subseteq} & & & \\
\mathcal{U}(\mathrm{pt}(L)) & \xrightarrow{\subseteq} & L^{\delta} & & & &
\end{array}$$

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