Probability via logic: semantic analysis and proof theory

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joint work (in progress) with
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Motivation: A probabilistic 2-layer logic (e.g. P. Baldi, P. Cintula, C. Noguera 2020)

$$A ::= p \mid \top \mid \bot \mid \neg A \mid A \sqcap A \mid A \sqcup A$$

$$\phi ::= \mu(A) \mid 1 \mid 0 \mid \sim \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \oplus \phi \mid \phi \ominus \phi$$

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- Classical logic axioms for the non-modal formulas
- $\oplus$ is associative, commutative, with 0 as neutral element
- $\oplus$ preserves all finite non-empty meets and joins
- $\oplus$ and $\ominus$ are residuals of each other

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$$\phi ::= \mu(A) \mid 1 \mid 0 \mid\sim \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \oplus \phi \mid \phi \ominus \phi$$

A1. From $A \vdash B$ infer $\mu(A) \vdash \mu(B)$;

A2. $\mu(\neg A) \dashv\vdash \sim \mu(A)$;

A3. $(\mu(A) \ominus \mu(A \wedge B)) \oplus \mu(B) \dashv\vdash \mu(A \vee B)$;

Nec. from $\top \vdash A$ infer $1 \vdash \mu(A)$.

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- Semantic framework:
  - Classical formulas are interpreted in a Boolean algebra $\mathbb{B}$.
  - Probability formulas are interpreted on an (MV-)algebra $\mathbb{C}$.
  - $\mu : \mathbb{B} \to \mathbb{C}$, a monotone map.

Proper Multi-type Display Calculi

- **Display property**:

$$\dfrac{\dfrac{Y \vdash X > Z}{X \,;\, Y \vdash Z}}{X \vdash Z < Y}$$

  display rules semantically justified by **adjunction/residuation**

- **Multi-type:** Separate **syntactic types** for different types of semantic objects
- **Proper:** Rules closed under **uniform substitution** (Wansing '98) within each type
- **Canonical proof of cut elimination (via metatheorem)**

Display calculi and correspondence

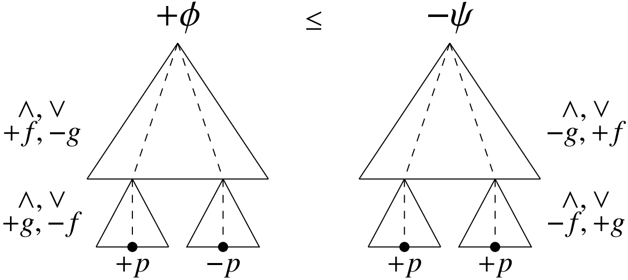
- The algorithm **ALBA** (properly adjusted) can transform an analytic inductive inequality into primitive quasi-inequalities.
- Analytic rules in display calculi semantically correspond to primitive quasi-inequalities.

Which logics are properly displayable?

[Kracht 96], [Ciabattoni[+]15], [Greco[+]16]

Complete characterization:

1. the logics of any **basic** normal (D)LE;
2. axiomatic extensions of these with **analytic inductive inequalities**:



The gaps

1. Many-sorted signature and heterogeneous connectives.
2. The connective μ is monotone not normal.
3. The connective $\oplus$ is regular (for join preservation).

Monotone modal logic as a 2-sorted frame

A monotone neighbourhood frame [Chellas 80], [Herzig+ 96], [Hansen 03]

$$\mathbb{N} := (W, \nu : W \to \mathcal{P}\mathcal{P}(W))$$

can be represented as a **2-sorted n-frame**:

$$\mathbb{K} := (X, Y, R_\nu, R_\ni, R_{\nu^c}, R_{\not\ni}) \qquad \text{where}$$

- $X := W$ and $Y := \mathcal{P}(W)$;
- $R_\nu \subseteq X \times Y \quad w\, R_\nu\, Z$ iff $Z \in \nu(w)$;
- $R_\ni \subseteq Y \times X \quad Z\, R_\ni\, w$ iff $w \in Z$ for all $x \in X$ and $Z \in Y$.

$\nabla\varphi := \langle\nu\rangle[\ni]\varphi$

$\mathbb{N}, w \Vdash \nabla\varphi$

iff $\exists Z\,(Z \in \nu(w)\ \&\ Z \subseteq \varphi^{\mathbb{N}})$

iff $\exists Z\,(w\, R_\nu\, Z\ \&\ \forall z\,(z \in Z \Rightarrow z \Vdash \varphi))$

iff $\exists Z\,(w\, R_\nu\, Z\ \&\ \forall z\,(Z\, R_\ni\, z \Rightarrow z \Vdash \varphi))$

iff $\exists Z\,(w\, R_\nu\, Z\ \&\ Z \Vdash [\ni]\varphi)$

iff $\mathbb{K}, w \Vdash \langle\nu\rangle[\ni]\varphi$

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- $X := W$ and $Y := \mathcal{P}(W)$;
- $R_{\nu^c} \subseteq X \times Y \quad w\, R_{\nu^c}\, Z$ iff $Z \notin \nu(w)$;
- $R_{\not\ni} \subseteq Y \times X \quad Z\, R_{\not\ni}\, w$ iff $w \notin Z$ for all $x \in X$ and $Z \in Y$.

$$\nabla\varphi := [\nu^c]\langle\not\ni\rangle\varphi$$

$\mathbb{N}, w \Vdash \nabla\varphi$

iff $\forall Z\, (Z \notin \nu(w) \;\Rightarrow\; \varphi^{\mathbb{N}} \nsubseteq Z)$

iff $\forall Z\, (w\, R_{\nu^c}\, Z \;\Rightarrow\; \exists z\, (z \notin Z \;\&\; z \in \varphi^{\mathbb{N}}))$

iff $\forall Z\, (w\, R_{\nu^c}\, Z \;\Rightarrow\; \exists z\, (Z\, R_{\not\ni}\, z \;\&\; z \in \varphi^{\mathbb{N}}))$

iff $\forall Z\, (w\, R_{\nu^c}\, Z \;\Rightarrow\; Z \Vdash \langle\not\ni\rangle\varphi)$

iff $\mathbb{K}, w \Vdash [\nu^c]\langle\not\ni\rangle\varphi$

Monotone modal logic as a 2-sorted frame

A monotone neighbourhood frame [Chellas 80], [Herzig+ 96], [Hansen 03]

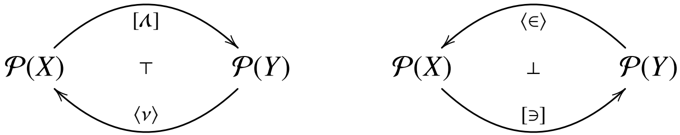
$$\mathbb{N} := (W, \nu : W \to \mathcal{P}\mathcal{P}(W))$$

can be represented as a **2-sorted n-frame**:

$$\mathbb{K} := (X, Y, R_\nu, R_\ni, R_{\nu^c}, R_{\not\ni})$$

and as a **heterogeneous m-algebra**:

$$\mathbb{H} := (\mathcal{P}(X), \mathcal{P}(Y), \langle\nu\rangle, [\ni], [\nu^c], \langle\not\ni\rangle)$$



- $\langle\nu\rangle$ and $[\ni]$ (resp. $[\nu^c]$ and $\langle\not\ni\rangle$) **multi-type normal operators**.

Monotone modal logic algebraically

Let $\mathbb{A}_1, \mathbb{A}_2$ be complete lattices and $\nabla : \mathbb{A}_1 \to \mathbb{A}_2$ be a monotone map. We define maps:

- $[\ni], \langle \not\ni \rangle : \mathbb{A}_1 \to \mathcal{P}(\mathbb{A}_1)$;
- $\langle \nu \rangle, [\nu^c] : \mathcal{P}(\mathbb{A}_1) \to \mathbb{A}_2$;

$$[\ni]a := \{b \in \mathbb{A} \mid b \leq a\} \qquad \langle \nu \rangle B := \bigvee \{\nabla b \mid b \in B\}$$

$$[\nu^c]B := \bigwedge \{\nabla b \mid b \notin B\} \qquad \langle \not\ni \rangle a := \{b \in \mathbb{A} \mid a \not\leq b\}.$$

Then $[\ni], \langle \not\ni \rangle, \langle \nu \rangle, [\nu^c]$ are **normal operators** and

$$\nabla a = \langle \nu \rangle [\ni] a = [\nu^c] \langle \not\ni \rangle a.$$

Positional translation

If $\mathbb{F}$ is a monotone n-frame, $\varphi \Rightarrow \psi$ is an $\mathcal{L}_\nabla$-sequent, $\mathbb{F}^\star$ its associated two-sorted n-frame, then

$$\mathbb{F} \Vdash \varphi \Rightarrow \psi \quad \text{iff} \quad \mathbb{F}^\star \Vdash \tau(\varphi \Rightarrow \psi).$$

$$\tau(\varphi \Rightarrow \psi) \ := \ \tau_1(\varphi) \vdash \tau_2(\psi)$$

$$\begin{array}{rcl@{\qquad}rcl}
\tau_1(p) & := & p & \tau_2(p) & := & p \\
\tau_1(\varphi \wedge \psi) & := & \tau_1(\varphi) \wedge \tau_1(\psi) & \tau_2(\varphi \wedge \psi) & := & \tau_2(\varphi) \wedge \tau_2(\psi) \\
\tau_1(\nabla\varphi) & := & \langle \nu \rangle [\ni] \tau_1(\varphi) & \tau_2(\nabla\varphi) & := & [\nu^c] \langle \not\ni \rangle \tau_2(\varphi)
\end{array}$$

- Positional translation allows us to transform **more** sequents into analytic inductive sequents.

ALBA rules for regular connectives (PSZ16)

- Adjunction rules (only for unary regular connectives):

$$\frac{f(\phi) \leq \psi}{f(\bot) \leq \psi \quad \phi \leq \blacksquare_f \psi}$$

- Approximation rules:

$$\frac{\mathbf{i} \leq f(\phi)}{[\mathbf{i} \leq f(\bot)] \quad ⅋ \quad [\mathbf{j} \leq \phi \quad \mathbf{i} \leq f(\mathbf{j})]}$$

$$\frac{\mathbf{i} \leq k(\overline{\phi}_{\epsilon_k^+}, \overline{\psi}_{\epsilon_k^-})}{\underset{P \subseteq \epsilon_k^+, N \subseteq \epsilon_k^-}{⅋} (\mathbf{i} \leq k(\overline{\mathbf{j}}_P, \overline{\bot}_{\epsilon_k^+ \setminus P}, \overline{\mathbf{m}}_N, \overline{\top}_{\epsilon_k^- \setminus N}) \ \& \ \underset{e \in P}{\&}(\mathbf{j}_e \leq \phi_e) \ \& \ \underset{e \in N}{\&}(\psi_e \leq \mathbf{m}_e))}$$

ABLA succeeds: But only when non-unary regular connectives appear exclusively in the skeleton.

Spelling out the approximation rule

We have:

$$\boldsymbol{i} \leq \psi_1 \oplus \psi_2 \quad \Leftrightarrow$$

- $[\boldsymbol{i} \leq 0 \oplus 0]$ OR
- $[\boldsymbol{i} \leq \boldsymbol{j}_1 \oplus 0 \ \& \ \boldsymbol{j}_1 \leq \psi_1]$ OR
- $[\boldsymbol{i} \leq 0 \oplus \boldsymbol{j}_2 \ \& \ \boldsymbol{j}_2 \leq \psi_2]$ OR
- $[\boldsymbol{i} \leq \boldsymbol{j}_1 \oplus \boldsymbol{j}_2 \ \& \ \boldsymbol{j}_1 \leq \psi_1 \ \& \ \boldsymbol{j}_2 \leq \psi_2]$.

An example

$$\forall[(p \ominus q) \oplus q \leq p \vee q]$$

iff $\forall[\mathbf{i} \leq (p \ominus q) \oplus q \;\&\; p \vee q \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]$

iff $\forall[\mathbf{i} \leq (\mathbf{j}_1 \ominus \mathbf{n}) \oplus \mathbf{j}_2 \;\&\; \mathbf{j}_1 \leq \mathbf{m} \;\&\; \mathbf{j}_2 \leq \mathbf{n} \;\&\; \mathbf{j}_2 \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]\&[\ldots]$

iff $\forall[\mathbf{j}_1 \leq \mathbf{m} \;\&\; \mathbf{j}_2 \leq \mathbf{n} \;\&\; \mathbf{j}_2 \leq \mathbf{m} \Rightarrow (\mathbf{j}_1 \ominus \mathbf{n}) \oplus \mathbf{j}_2 \leq \mathbf{m}]\&[\ldots]$

Which yields the following **structural rule**:

$$\text{Ł3}\ \frac{X_1 \vdash Y_1 \qquad X_2 \vdash Y_2 \qquad X_2 \vdash Y_3}{(X_1 \hat{\ominus} Y_2) \hat{\oplus} X_2 \vdash Y_1 \check{\vee} Y_3}$$

The red bracket

- We have 3 cases:
  1. $\boldsymbol{i} \leq 0 \oplus 0 \;\&\; p \vee q \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}$.
  2. $\boldsymbol{i} \leq j_2 \;\&\; j_2 \leq q \;\&\; p \vee q \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}$.
  3. $\boldsymbol{i} \leq j_1 \;\&\; j_1 \leq p \ominus q \;\&\; p \vee q \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}$.
- All 3 cases are tautological statements.

Putting everything together

$$A ::= p \mid \top \mid \bot \mid \neg A \mid A \sqcap A \mid A \sqcup A$$

$$\alpha ::= [\ni]A \mid \langle \not\ni \rangle A$$

$$\phi ::= \langle \nu \rangle \alpha \mid [\nu^c]\alpha \mid 1 \mid 0 \mid \sim \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \oplus \phi \mid \phi \ominus \phi$$

- A1. From $A \vdash B$ infer $\langle \nu \rangle [\ni] A \vdash [\nu^c]\langle \not\ni \rangle B$;
- A2. $\langle \nu \rangle [\ni](\neg A) \vdash \sim \langle \nu \rangle [\ni] A$ and $\sim [\nu^c]\langle \not\ni \rangle A \vdash [\nu^c]\langle \not\ni \rangle \neg A$;
- A3a. $(\langle \nu \rangle [\ni] A \ominus [\nu^c]\langle \not\ni \rangle (A \wedge B)) \oplus \langle \nu \rangle [\ni] B \vdash [\nu^c]\langle \not\ni \rangle (A \vee B)$;
- A3b. $\langle \nu \rangle [\ni](A \vee B) \vdash ([\nu^c]\langle \not\ni \rangle A \ominus \langle \nu \rangle [\ni](A \wedge B)) \oplus [\nu^c]\langle \not\ni \rangle B$;
- Nec. from $\top \vdash A$ infer $1 \vdash [\nu^c]\langle \not\ni \rangle A$.

Structural rules

- A1.

$$\text{M}\ \frac{\langle\hat{\not\ni}\rangle\langle\hat{\in}\rangle\Gamma \vdash \Delta}{\langle\hat{\lambda}^c\rangle\langle\hat{\nu}\rangle\Gamma \vdash \Delta}$$

- A3a.

$$\frac{\langle\hat{\not\ni}\rangle(\langle\hat{\in}\rangle X \mathbin{\hat{\sqcap}} \langle\hat{\in}\rangle Y) \vdash Z \qquad \langle\hat{\not\ni}\rangle\langle\hat{\in}\rangle X \vdash W \qquad \langle\hat{\not\ni}\rangle\langle\hat{\in}\rangle Y \vdash W}{\langle\hat{\nu}\rangle X \mathbin{\hat{\oplus}} (\langle\hat{\nu}\rangle Y \mathbin{\hat{\ominus}} [\check{\nu}^c]Z) \vdash [\check{\nu}^c]W}$$

- Nec.

$$\text{N}\ \frac{\langle\hat{\not\ni}\rangle\hat{\top} \vdash \Gamma}{\hat{1} \vdash [\check{\nu}^c]\Gamma}$$

Conclusions

- Proof system for probabilistic logics
- Modular tools to tackle the problems
- What about cut elimination?