Graded modal logic with a single modality

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Graded modal logic: Syntax and Semantics

$$\Phi_G \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond_n \varphi, \quad n \in \mathbb{N}^+.$$

$$(W, R, v), x \models \Diamond_n \varphi \quad \text{iff} \quad |\{y \in R[x] : M, y \models \varphi\}| \geq n$$

- Shorthand: $\Diamond_{!n}\varphi := \Diamond_n \varphi \wedge \neg\Diamond_{n+1}\varphi$

Axioms:

- Classical tautologies;
- $\Diamond_n \bot \to \bot$;
- $\Diamond_{n+1}\varphi \to \Diamond_n \varphi$;
- $\Box_1(\varphi \to \psi) \to (\Diamond_n \varphi \to \Diamond_n \psi)$;
- $\neg\Diamond_1(\varphi \wedge \psi) \wedge \Diamond_{!m}\varphi \wedge \Diamond_{!n}\psi \to \Diamond_{!(m+n)}(\varphi \vee \psi)$;
- Modus ponens and Necessitation (for $\Box_1$).

Graded modal logic: Properties and completeness

$$\Phi_G \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond_n \varphi, \quad n \in \mathbb{N}^+.$$

$$(W, R, v), x \models \Diamond_n \varphi \quad \text{iff} \quad |\{y \in R[x] : M, y \models \varphi\}| \geq n$$

Properties:

- All $\Diamond_n$ are monotone but not join-preserving.
- The logic is decidable.
- Strong finite model property.

Completeness:

- For any ultrafilter u and $\varphi \in \Phi_G$ define $e_u(\varphi) = \sup\{n \in \mathbb{N} \mid \Diamond_n \varphi \in u\}$.
- For any ultrafilters u, w define $e(w, u) = \min\{e_u(\varphi) \mid \varphi \in w\}$;
- $(\mathrm{Uf} \times \mathbb{N}, R)$ where $(u, k)R(w, m)$ if and only if $e(w, u) \geq m$.

Graded modal logic with a single modality

$$\Phi \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi.$$

$$(W, R, v), x \models_n \Diamond\varphi \quad \text{iff} \quad |\{y \in R[x] : M, y \models \varphi\}| \geq n$$

Questions:

- Does the logic depend on n?
- What are the logics $\mathcal{L}_n$?
- Can we axiomatize them?

Some properties using graded modal logic

- $\epsilon : \mathbb{N}^{+} \times \Phi \to \Phi_G$ recursively $\epsilon(n, \Diamond\varphi) = \Diamond_n \epsilon(n, \varphi)$.
- Each logic $\mathcal{L}_n$ is decidable.
- Each logic $\mathcal{L}_n$ has the strong finite model property.

A first observation

$$(\Diamond(\psi \vee (p \wedge \neg q)) \wedge \neg\Diamond\psi) \wedge (\Diamond(\sigma \vee (q \wedge \neg p)) \wedge \neg\Diamond\sigma) \rightarrow \Diamond(p \vee q)$$

A first observation

$$(\Diamond(\psi \vee (p \wedge \neg q)) \wedge \neg\Diamond\psi) \wedge \big(\Diamond(\sigma \vee (q \wedge \neg p)) \wedge \neg\Diamond\sigma\big) \to \Diamond(p \vee q)$$

A first observation

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- Shorthand: $\Diamond_1^{\psi}\varphi := \Diamond(\psi \vee \varphi) \wedge \neg\Diamond\psi$

  $$M, x \models_n \Diamond_1^{\psi}\varphi \quad \Rightarrow \quad \exists y \in W, xRy \;\&\; M, y \models_n \varphi$$

- Convention: Greek letters α, β, γ denote sequences of mutually contradictory formulas. When we write $\alpha_1, \ldots, \alpha_n$ it means $\alpha_i \wedge \alpha_j \to \bot$ is provable in classical logic for $i \neq j$.

Relating the logics $\mathcal{L}_n$

$$\zeta_n := \left(\bigwedge_{i=1}^{n} \diamond_1^{q_i}\alpha_i\right) \to \diamond\left(\bigvee_{i=1}^{n} \alpha_i\right).$$

Lemma

If $n < m$ then $\zeta_n \in \mathcal{L}_n$ but $\zeta_n \notin \mathcal{L}_m$.

Relating the logics $\mathcal{L}_n$

$$\zeta_n \coloneqq \left(\bigwedge_{i=1}^{n} \Diamond_1^{q_i}\alpha_i\right) \to \Diamond\left(\bigvee_{i=1}^{n} \alpha_i\right).$$

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If $n < m$ then $\zeta_n \in \mathcal{L}_n$ but $\zeta_n \notin \mathcal{L}_m$.

Lemma

Assume $n < m$ such that $m - 1 = (n - 1) \cdot k + r$ where $r < n - 1$.
Then, $\mathcal{L}_m \subseteq \mathcal{L}_n$ if and only if $r < k$.

Relating the logics $\mathcal{L}_n$

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$$\theta_n := \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n-1} \left(\Diamond_1^{q_i^j} \alpha_i^j \wedge \bigwedge_{i=1}^{n} \neg\Diamond(\bigvee_{j=1}^{n-1} \alpha_i^j)\right) \to \bigvee_{s:n\to n-1} \neg\Diamond \bigvee_{i=1}^{n} \alpha_i^{s(i)}.$$

Let's talk completeness

The basics for all $\mathcal{L}_n$

- Classical tautologies and Modus Ponens;
- $\Diamond\bot \to \bot$;
- $\vdash p \to q \; / \; \vdash \Box p \to \Box q$.

Key Lemma

Let $(\mathbb{B}, \Diamond)$ be a Boolean algebra with a monotone operation satisfying $\Diamond\bot = \bot$. Let u be an ultrafitler on $\mathbb{B}$ and let

$$Z_u = \{a \in \mathbb{B} \mid \forall b \in \mathbb{B}(\Diamond_1^b a \notin u)\}.$$

Then Z_u is an ideal on $\mathbb{B}$ such that $\Diamond a \in u$ implies that $a \notin Z_u$.

Completeness for $\mathcal{L}_2$

Added Axiom

- $\zeta_2 \;=\; \left[\Diamond_1^{q_1}(\alpha_1) \wedge \Diamond_1^{q_2}(\alpha_2)\right] \rightarrow \Diamond(\alpha_1 \vee \alpha_2)$

$$e_u(\varphi) = \begin{cases} 2 & \text{if } \Diamond\varphi \in u, \\ 1 & \text{if } \Diamond\varphi \notin u \text{ and } (\exists\psi)(\Diamond_1^{\psi}\varphi \in u), \\ 0 & \text{otherwise.} \end{cases}$$

- Notice that $e_u^{-1}[0] = Z_u$.
- $e(w,u) = \min\{e_u(\varphi) \mid \varphi \in w\}$;
- $(\text{Uf} \times \{1,2\}, R)$ where $(u,k)R(w,m)$ if and only if $e(w,u) \geq m$.

Completeness for $\mathcal{L}_3$

Added Axiom

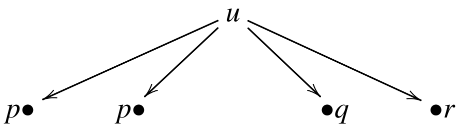
- $\zeta_3 \; = \; \left[\Diamond_1^{q_1}(\alpha_1) \wedge \Diamond_1^{q_2}(\alpha_2) \wedge \Diamond_1^{q_3}(\alpha_3)\right] \rightarrow \Diamond(\alpha_1 \vee \alpha_2 \vee \alpha_3)$
- $\left[\Diamond_1^{q_1}(\alpha_2) \wedge \Diamond_1^{q_2}(\beta_2) \wedge \Diamond(\alpha_1 \vee \beta_1) \wedge \neg\Diamond(\alpha_1 \vee \alpha_2)\right] \rightarrow \Diamond(\beta_1 \vee \beta_2)$

$$e_u(\varphi) = \begin{cases} 3 & \text{if } \Diamond\varphi \in u, \\ 2 & \text{if } e_u(\varphi) \neq 3 \text{ and } (\forall\psi)(\Diamond_1^{\sigma}\psi \in u, \varphi \wedge \psi = \bot \implies \Diamond(\varphi \vee \psi) \in u) \\ 1 & \text{if } e_u(\varphi) \notin \{3, 2\} \text{ and } (\exists\psi)(\Diamond_1^{\psi}\varphi \in u), \\ 0 & \text{otherwise.} \end{cases}$$

- Notice that $e_u^{-1}[0] = Z_u$.
- $e(w, u) = \min\{e_u(\varphi) \mid \varphi \in w\}$;
- $(\mathrm{Uf} \times \{1, 2, 3\}, R)$ where $(u, k)R(w, m)$ if and only if $e(w, u) \geq m$.

Completeness for $\mathcal{L}_4$

Completeness for $\mathcal{L}_4$?



- For any permuation of $\{p, q, r\}$, π, $\mathrm{Th}(u) = \pi(\mathrm{Th}(u))$.
- If a uniform way to define e_u for $\mathcal{L}_4$ exists, then $e_u(p) = e_u(q) = e_u(r)$.
- For $n \geq 4$, if e_u exists, we need to make arbitrary choices.

Some arithmetic?

$$\bigwedge_{j\in J}(\sum_{i\in I} x_i^j < n) \to \bigvee_{k\in K\subset J}(\sum_{h\in H\subset I} x_h^k < n)$$

$$\Downarrow$$

$$\left(\Diamond_1^{q_i^j}(\alpha_i^j) \wedge \bigwedge_{j\in J}\neg\Diamond\bigvee_{i\in I}\alpha_i^j\right) \to \bigvee_{k\in K\subset J}\neg\Diamond\bigvee_{h\in H\subset I}\alpha_h^k$$

Completeness for $\mathcal{L}_n$?

- We want a map $e_u : \Phi \to \mathbb{N}$ such that:

1. $e_u(\varphi) \geq n$ if $\Diamond(\varphi) \in u$,
2. $e_u(\varphi) < n$ if $\Diamond(\varphi) \notin u$,
3. $e_u(\varphi) + e_u(\psi) = e_u(\varphi \wedge \psi) + e_u(\varphi \vee \psi)$, for every $\psi \in \Phi$,
4. $e_u(\varphi) = 0$ if for every $\psi \in \Phi$, $\Diamond_1^{\psi}\varphi \notin u$.

Lemma

Such an e_u exists for every ultrafilter u of the algebra of $\mathcal{L}_n$.

- Notice that $e_u^{-1}[0] = Z_u$.
- $e(w, u) = \min\{e_u(\varphi) \mid \varphi \in w\}$;
- $(\mathsf{Uf} \times n, R)$ where $(u, k)R(w, m)$ if and only if $e(w, u) \geq m$.

Conclusions

- Identified fragments of graded modal logic.
- For each $n \in \mathbb{N}$ a different logic $\mathcal{L}_n$.
- Some completeness results. Can we do better?