Difference-restriction algebras of partial functions: axiomatizations and representations

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based on joint work with Brett McLean

Centro de Matemática da Universidade de Coimbra

Topology, **A**lgebra, and **C**ategories in **L**ogic

Coimbra, June 20-24, 2022

What is a partial function?

Let X be a set.

- A partial function on X is a subset $f \subseteq X \times X$ such that

$$(x, y) \in f \text{ and } (x, z) \in f \implies y = z.$$

  We write $f(x) = y$ if $(x, y) \in f$.

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- There are many different operations one can consider on the set $\mathcal{PF}(X)$ of all partial functions on X...

What is an algebra of partial functions?

We will consider universal algebras of the signature $\{-, \triangleright\}$.

An algebra of partial functions is a $\{-, \triangleright\}$-algebra whose universe is contained in $\mathcal{PF}(X)$, and whose operations are interpreted as follows:

Relative complement: $f - g := \{(x, y) \mid (x, y) \in f \text{ and } (x, y) \notin g\}$.

Domain restriction: $f \triangleright g := \{(x, y) \mid x \in \mathrm{dom}(f) \text{ and } (x, y) \in g\}$.

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- Able to express intersection: $f \cdot g = f - (f - g)$.
- Well-behaved order-structure: $f \leq g \iff f \cdot g = f \iff f \subseteq g$.
- Able to compare domains: $\mathrm{dom}(f) \subseteq \mathrm{dom}(g) \iff f \leq g \triangleright f$.

Algebras representable by partial functions

Let $\mathcal{A}$ be a algebra of the signature $\{-, \triangleright\}$. We say that $\mathcal{A}$ is representable if it is isomorphic to an algebra of partial functions.

Notation: For $a, b \in \mathcal{A}$, denote $a \cdot b := a - (a - b)$.

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Some axioms

(Ax.1) $a - (b - a) = a$

(Ax.2) $a \cdot b = b \cdot a$

(Ax.3) $(a - b) - c = (a - c) - b$

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A restriction-semilattice is a $\{\cdot, \triangleright\}$-algebra such that $(\mathcal{A}, \cdot)$ is a semillatice an $\mathcal{A}$ satisfies (Ax.4)-(Ax.5).

Subtraction algebras and Boolean algebras

Recall: $(\mathcal{A}, -)$ is a *subtraction algebra* if it satisfies (Ax.1)-(Ax.3):

(Ax.1) $a - (b - a) = a$ **(Ax.2)** $a \cdot b = b \cdot a$ **(Ax.3)** $(a-b)-c = (a-c)-b$

Proposition (Schein 1992; B., McLean 2020)

- If $\mathcal{A}$ is a subtraction algebra, then the set $a^{\downarrow} := \{b \in \mathcal{A} \mid b \leq a\}$ is a Boolean algebra. The complement of $b \in a^{\downarrow}$ is $\overline{b} := a - b$.

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- If $(\mathcal{A}, \cdot)$ is a semilattice with zero 0 and, for every $a \in \mathcal{A}$, we have a Boolean algebra $(a^{\downarrow}, \cdot, 0, a, \overline{(\_)}^{a})$, then setting $a - b := \overline{(a \cdot b)}^{a}$ defines a subtraction algebra structure on $\mathcal{A}$ (on which $a \cdot b = a - (a - b)$ is a valid equation).

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Domain inclusion - definition

$(\mathcal{A}, \cdot, \rhd)$ - restriction-semilattice

- $(\mathcal{A}, \cdot)$ is a semilattice,
- $\mathcal{A}$ satisfies (Ax.4) and (Ax.5):

$$\text{(Ax.4)} \quad (a \rhd c) \cdot (b \rhd c) = (a \rhd b) \rhd c$$
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We define the relation $\preceq$ on $\mathcal{A}$ by

$$a \preceq b \iff a \leq (b \rhd a)$$

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  (the induced equivalence relation will be denoted by $\sim$)
- The relations $\preceq$ and $\leq$ coincide on each poset $a^{\downarrow} := \{b \in \mathcal{A} \mid b \leq a\}$.

Filters

Definition

A filter on a semilattice $(\mathcal{A}, \cdot)$ is a nonempty subset $F \subseteq \mathcal{A}$ such that

- F is upward closed: $a \in F$ and $a \leq b$ implies $b \in F$,
- F is closed under binary meets: $a, b \in F$ implies $a \cdot b \in F$.

Remark: There is an order-embedding $(\mathcal{A}, \leq) \hookrightarrow (\mathrm{Filt}(\mathcal{A}), \supseteq)$,

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If $\mathcal{A}$ is a restriction-semilattice, then the operations $\cdot$ and $\triangleright$ may be extended to operations $\cdot_F$ and $\triangleright_F$ on $\mathrm{Filt}(\mathcal{A})$ as follows:

$$F \cdot_F G := \langle F \cdot G \rangle_{\mathsf{Filt}} \quad \text{and} \quad F \triangleright_F G := \langle F \triangleright G \rangle_{\mathsf{Filt}}.$$

(where $F \cdot G := \{a \cdot b \mid a \in F,\ b \in G\}$ and $F \triangleright G := \{a \triangleright b \mid a \in F,\ b \in G\}$)

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If $\mathcal{A}$ is a restriction-semilattice, then so is $(\mathrm{Filt}(\mathcal{A}), \cdot_F, \rhd_F)$.

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In particular, the *"domain containment"* relation $\preceq$ on $\mathcal{A}$ extends to $\mathrm{Filt}(\mathcal{A})$, and we have:

- The relation $\preceq$ on $\mathrm{Filt}(\mathcal{A})$ is a partial order that contains $\supseteq$.
- The relations $\preceq$ and $\supseteq$ coincide on each upset $F^{\uparrow} := \{G \in \mathrm{Filt}(\mathcal{A}) \mid F \subseteq G\}$.

Maximal filters

A filter is maximal if it is proper and $\subseteq$-maximal among all proper filters.

Proposition

Let $\mathcal{A}$ be a subtraction algebra, $a \in \mathcal{A}$. There is a bijection between the maximal filters of $a^{\downarrow}$ and the maximal filters of $\mathcal{A}$ that contain a:

- if $F \subseteq \mathcal{A}$ is maximal and $a \in F$, then $F \cap a^{\downarrow}$ is a maximal filter of $a^{\downarrow}$,
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- *A filter F is maximal if and only if for all $a \in F$, $b \in \mathcal{A}$, precisely one of $a \cdot b$ and $a - b$ belongs to F.*

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Proof:

$$F \text{ maximal} \iff F \cap a^{\downarrow} \text{ maximal} \iff a \cdot b \in F \cap a^{\downarrow} \quad \vee \quad \overline{a \cdot b}^{a^{\downarrow}} \in F \cap a^{\downarrow}.$$

Representation theorems

A representation theorem for difference-restriction algebras

Representation of difference-restriction algebras

Let $\mathcal{A}$ be a difference-restriction algebra. Then, the assignment

$$\theta : \mathcal{A} \to \mathcal{PF}(\mathrm{Filt}_{\max}(\mathcal{A})), \qquad a \mapsto a^\theta := \{(H, F) \mid H \sim F \text{ and } a \in F\}$$

is a representation of $\mathcal{A}$ by partial functions.

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Remark: If $\mathcal{A}$ has a top element, that is, if $\mathcal{A}$ is a Boolean algebra, then θ gives the classical representation of a Boolean algebra as a field of sets.

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- $(H, F), (H, G) \in a^\theta \implies F \sim G$ and $F, G \in (a^\uparrow)^\uparrow \implies F = G$.

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$$\iff (H, F) \in (a - b)^\theta.$$

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Theorem

The class of $\{-, \triangleright\}$-algebras representable by partial functions is a variety, axiomatised by the finite set of equations (Ax.1) – (Ax.5).

Complete representability

A representation $(\theta : \mathcal{A} \hookrightarrow \mathcal{PF}(X),\ a \mapsto a^\theta)$ is:

- meet complete if for every *nonempty* subset $S \subseteq \mathcal{A}$, if $\inf S$ exists then
$$\theta(\inf S) = \bigcap \theta[S],$$
- join complete if for every subset $S \subseteq \mathcal{A}$, if $\sup S$ exists, then
$$\theta(\sup S) = \bigcup \theta[S],$$
- atomic if whenever $(x, y) \in a^\theta$, there is an atom z such that $(x, y) \in z^\theta$.

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Proposition

The following are equivalent:

- θ is meet complete,
- θ is join complete,
- θ is atomic.

Complete representability

We say that $\mathcal{A}$ is atomic if every nonzero element of $\mathcal{A}$ is greater than or equal to an atom.

Corollary

If $\mathcal{A}$ has a complete representation then $\mathcal{A}$ is atomic.

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- If $a \neq 0$, then $a^\theta \neq \emptyset$. Pick $(x, y) \in a^\theta$.

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- θ complete $\iff$ θ atomic $\implies$ $(x, y) \in z^\theta$ for some atom z.

Complete representability

We say that $\mathcal{A}$ is atomic if every nonzero element of $\mathcal{A}$ is greater than or equal to an atom.

Corollary

If $\mathcal{A}$ has a complete representation then $\mathcal{A}$ is atomic.

Proof: Let $\theta : \mathcal{A} \hookrightarrow \mathcal{PF}(X),\ a \mapsto a^\theta$ be a complete representation.

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- θ complete $\iff$ θ atomic $\implies$ $(x, y) \in z^\theta$ for some atom z.
- $(x, y) \in a^\theta \cap z^\theta = (a \cdot z)^\theta \implies (a \cdot z)^\theta \neq \emptyset \implies a \cdot z \neq 0 \implies z < a$ $\square$

Complete representability

Proposition

Let $\mathcal{A}$ be an atomic difference-restriction algebra. Then, the assignment

$$\theta : \mathcal{A} \to \mathcal{PF}(\mathrm{At}(\mathcal{A})), \qquad a \mapsto a^\theta := \{(x, y) \mid \; x \sim y \; \text{ and } \; y \leq a\}$$

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Theorem

The class of $\{-, \triangleright\}$-algebras that are completely representable by partial functions is axiomatised by the finite set of equations (Ax.1) – (Ax.5) together with the $\forall\exists\forall$ first-order formula stating that the algebra is atomic.

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The class of $\{-, \triangleright\}$-algebras that are completely representable by partial functions is not axiomatisable by any $\exists\forall\exists$ first-order theory.

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- If $(\mathcal{B}, 0, 1, \wedge, \neg)$ is a Boolean algebra, then setting
$$a - b := a \wedge \neg b \qquad \text{and} \qquad a \triangleright b := a \wedge b$$
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- $\mathcal{B}$ is atomic as a Boolean algebra if and only if it is atomic as a difference-restriction algebra (the underlying posets are the same),
- there are Boolean algebras $\mathcal{B}$ and $\mathcal{B}'$ that satisfy the same $\exists\forall\exists$ first-order theory, with $\mathcal{B}$ atomic and $\mathcal{B}'$ not atomic (McLean 2017). $\square$

Thank you for your attention!