On presheaf submonads of quantale enriched categories

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23/06/2022

(this talk is based on joint work with Maria Manuel Clementino)

Relations

A **relation** r from a set X to a set Y, gives us a way of discerning which elements $x \in X$ and $y \in Y$ are r-related, which is usually denoted by $x \; r \; y$.

We may present a relation r as a subset or as a two-valued map:

$$r \subseteq X \times Y, \qquad r : X \times Y \to \mathbf{2} = \{\bot, \top\}$$

$x \; r \; y$ may then be written as

$$(x, y) \in r, \qquad r(x, y) = \top.$$

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We will write $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ for a r a relation from X to Y. We may compose r with $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ using the usual relational composition:

$$x\ (s \cdot r)\ z \ \Leftrightarrow\ \exists y \in Y\ (x\ r\ y\ \&\ y\ s\ z).$$

Equivalently, this may be written as

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \wedge s(y, z).$$

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Orders

An **(pre)order** on a set X is a relation $a : X \looparrowright X$ such that,

$$a \cdot a \leq a \quad \text{and} \quad 1_X \leq a.$$

That is, a is a transitive and reflexive relation on X. This may be written as

$$x \leq y \ \& \ y \leq z \Rightarrow x \leq z \quad \text{and} \quad x \leq x,$$

or equivalently, when presenting r as a map $r : X \times X \to \mathbf{2}$

$$a(x,y) \wedge a(y,z) \leq a(x,z) \quad \text{and} \quad \top \leq a(x,x).$$

There is a clear similarity with these requirements and those of a *(quasi)metric* $a : X \times X \to [0,\infty]$ on X :

$$a(x,y) + a(y,z) \geq a(x,z) \quad \text{and} \quad 0 \geq a(x,x).$$

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In the previous example we replaced **2** with $[0,\infty]_+$.

The generalized framework encompassing both examples that we are interested is provided by a **quantale**.

A quantale $V = (V, \leq, \otimes, k)$ is a complete lattice equipped with a binary operation $\otimes$ (instead of $\wedge$ or $+$) respecting arbitrary joins in each variable and a $\otimes$-neutral element k (instead of $\top$ or 0).

As a category V is a "thin" symmetric monoidal-closed category, with internal homs $\hom(v, w)$ determined by

$$u \leq \hom(v, w) \;\Leftrightarrow\; u \otimes v \leq w.$$

A few examples:

- $\mathbf{2} = (\{\bot, \top\}, \leq, \wedge, \top)$;
- The powerset $(\mathcal{P}X, \subseteq, \cap, X)$, for any set X.
- The Lawvere quantale $[0,\infty]_+ = ([0,\infty], \geq, +, 0)$;
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V-relations

As we just seen, a relation $r : X \nrightarrow Y$ is a map $X \times Y \to \mathbf{2}$.

A V-**relation** $r : X \nrightarrow Y$ is a map $X \times Y \to V$.

Given two V-relations $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$ their composition $s \cdot r : X \nrightarrow Z$ is given by

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

Sets and V-relations define the category V-**Rel**.

The order from V induces an order on V-relations $r, r' : X \nrightarrow Y$,

$$r \leq r' \quad \Leftrightarrow \quad \forall x \in X,\ y \in Y : r(x, y) \leq r'(x, y),$$

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V-**Rel**

V-**Rel** also comes with an involution $(-)^\circ$ defined by

$$r^\circ(y,x) = r(x,y),$$

for any $x \in X$, $y \in Y$.

Each map $f : X \to Y$ induces the V-relation $f_\circ : X \nrightarrow Y$

$$f_\circ(x,y) = \begin{cases} k & \text{if } f(x) = y \\ \bot & \text{otherwise.} \end{cases}$$

Moreover, for every map $f : X \to Y$ we have that

$$f_\circ \cdot f^\circ \leq 1_Y \qquad \text{and} \qquad 1_X \leq f^\circ \cdot f_\circ,$$

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A V-**category** (X, a) is a set X equipped with a V-relation $a : X \times X \to V$, such that

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Equivalently

$$a(x, y) \otimes a(y, z) \leq a(x, z) \quad \text{and} \quad k \leq a(x, x)$$

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V-**Cat**

A morphism in **Ord** $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ satisfying

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for all $x, y \in X$.

A V-**functor** $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ satisfying

$$a(x, y) \leq b(f(x), f(y)),$$

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A V-functor is called **fully faithful** if the inequality is also an equality.

V-categories and V-functors define the category V-**Cat**.

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V-Cat

The quantale V is itself a V-category when equipped with its internal hom.

Given a V-category (X, a) its **opposite** V-category is $(X, a)^{op} = (X, a^\circ)$.

The order from V induces an order in each V-category (X, a) as follows

$$x \leq y \Leftrightarrow k \leq a(x, y).$$

This order makes every V-functor monotone.

Now, given V-categories (X, a), (Y, b), we order the hom-set V-**Cat**(X, Y) pointwise using the order inherited from Y :

$$f \leq g \Leftrightarrow f(x) \leq g(x) \quad \forall x \in X,$$

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V-**Cat** is a symmetric monoidal-closed category, with the unit given by $E = (\{*\}, k)$, where $k(*, *) = k$.

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V-**Dist**

Given two V-categories (X, a), (Y, b), a V-**distributor** $\psi : X \multimapdotinv Y$ is a V-relation $\psi : X \nrightarrow Y$ satisfying

$$b \cdot \psi \cdot a \leq \psi.$$

Equivalently

$$a(x', x) \otimes \psi(x, y) \otimes b(y, y') \leq \psi(x', y'),$$

holds for any $x, x' \in X$, $y, y' \in Y$.

Under relational composition, V-categories and V-distributors form the category V-**Dist**.
In V-**Dist**, the identity morphism for each V-category is its structure.

Moreover, V-**Dist** inherits the 2-categorical structure of V-**Rel**.

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$V\text{-}\mathbf{Dist}(X, Y) \cong V\text{-}\mathbf{Cat}(X^{\mathrm{op}} \otimes Y, V)$

Given two V-categories (X, a), (Y, b) and a V-relation $\varphi : X \nrightarrow Y$ the following are equivalent:

- $\varphi : (X, a) \multimap\!\!\rightarrow (Y, b)$ is a V-distributor;
- $\varphi : (X, a)^{\mathrm{op}} \otimes (Y, b) \to (V, \hom)$ is a V-functor.

This allows us to define a V-category structure on $V\text{-}\mathbf{Dist}(X, Y)$ by using the structure of $V\text{-}\mathbf{Cat}(X^{\mathrm{op}} \otimes Y, V)$.

Note that, in particular

$$V\text{-}\mathbf{Dist}(X, E) \cong V\text{-}\mathbf{Cat}(X^{\mathrm{op}} \otimes E, V) \cong V\text{-}\mathbf{Cat}(X^{\mathrm{op}}, V).$$

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Every V-functor $f:(X,a)\to(Y,b)$ induces a pair of V-distributors

$$f_* = b\cdot f_\circ : X \;{\circ\!\!\!\rightarrow}\; Y \qquad \text{and} \qquad f^* = f^\circ \cdot b : Y \;{\circ\!\!\!\rightarrow}\; X.$$

Moreover, we have the 2-functors

$$(-)_* : V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Dist}^{\mathrm{co}} \qquad \text{and} \qquad (-)^* : V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Dist}^{\mathrm{op}},$$

which map objects identically.

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Presheaf monad

The adjunction

$$V\text{-}\mathbf{Cat} \underset{V\text{-}\mathbf{Dist}(-,E)}{\overset{(-)^*}{\rightleftarrows}} V\text{-}\mathbf{Dist}^{\mathrm{op}} \qquad (\bot)$$

induces the
presheaf monad $\mathbb{P} = (P, \mathfrak{m}, \mathfrak{y})$ on V-**Cat**, where

- $PX = V\text{-}\mathbf{Cat}(X^{\mathrm{op}}, V) \cong V\text{-}\mathbf{Dist}(X, E)$.
- $P(f)(\varphi) = V\text{-}\mathbf{Dist}(f^*, E)(\varphi) = \varphi \cdot f^* = Y \overset{f^*}{\multimap} X \overset{\varphi}{\multimap} E$;
- $\mathfrak{y}_X(x) = a(-, x)$;
- $\mathfrak{m}_X(\Psi) = \Psi \cdot (\mathfrak{y}_X)_* = X \overset{(\mathfrak{y}_X)_*}{\multimap} PX \overset{\Psi}{\multimap} E$.

for any $x \in X$, $f : (X, a) \to (Y, b)$, $\varphi \in PX$ and $\Psi \in PPX$.

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Presheaf monad

Since

$$\begin{aligned} V\text{-}\mathbf{Dist}(X,Y) &\cong V\text{-}\mathbf{Cat}(X^{\mathrm{op}}\otimes Y, V) \\ &\cong V\text{-}\mathbf{Cat}(Y, V\text{-}\mathbf{Cat}(X^{\mathrm{op}}, V)) \\ &= V\text{-}\mathbf{Cat}(Y, PX) \end{aligned}$$

it follows that any V-distributor $X \mathrel{-\!\!\circ\!\!\to} Y$ can be seen as a morphism $Y \to X$ in the Kleisli category $\mathsf{Kl}(\mathbb{P})$.

In fact, we have that $V\text{-}\mathbf{Dist} \cong \mathsf{Kl}(\mathbb{P})$.

This parallels nicely the fact that $V\text{-}\mathbf{Rel} \cong \mathsf{Kl}(\mathbb{P}_d)$, where $\mathbb{P}_d$ is the discrete presheaf monad on **Set**.

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A new Beck-Chevalley type condition

A commutative square in **Set**

$$\begin{array}{ccc} W & \xrightarrow{\;l\;} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[\;f\;]{} & Y \end{array}$$

is said to be a (BC)-**square** if the following diagram commutes in **Rel**

$$\begin{array}{ccc} W & \overset{l_\circ}{\relbar\!\!\!\mapsto\!\!\!\rightarrow} & Z \\ {\scriptstyle g^\circ}\uparrow & & \uparrow{\scriptstyle h^\circ} \\ X & \underset{f_\circ}{\relbar\!\!\!\mapsto\!\!\!\rightarrow} & Y \end{array}$$

A new Beck-Chevalley type condition

A commutative square in V-**Cat**

$$\begin{array}{ccc} (W,d) & \xrightarrow{\;l\;} & (Z,c) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ (X,a) & \xrightarrow[f]{} & (Y,b) \end{array}$$

is said to be a $(BC)^*$-**square** if the following diagram commutes in V-**Dist**

$$\begin{array}{ccc} (W,d) & \xrightarrow{\;l_*\;} & (Z,c) \\ {\scriptstyle g^*}\uparrow & & \uparrow{\scriptstyle h^*} \\ (X,a) & \xrightarrow[f_*]{} & (Y,b) \end{array}$$

(In fact, it's enough to verify that if $h^* \cdot f_* \leq l_* \cdot g^*$)

A new Beck-Chevalley type condition

It's well known that a map $f : X \to Y$ is a monomorphism if and only if

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a (BC)-square.

In parallel with this, a V-functor $(X, a) \to (Y, b)$ being fully faithful is equivalent to

$$\begin{array}{ccc} (X, a) & \xrightarrow{1} & (X, a) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle f} \\ (X, a) & \xrightarrow[f]{} & (Y, b) \end{array}$$

being a $(BC)^*$-square.

$(BC)^*$ functors and $(BC)^*$ natural transformations

Consider the following:

- A **Set**-endofunctor is said to satisfy (BC) if it preserves (BC)-squares.
- A natural transformation $\alpha : T \to T'$ between **Set**-endofunctors satisfies (BC) if, for each morphism in **Set**, its naturality square is a (BC)-square.

Analogously, we define:

- A V-**Cat**-endofunctor is said to **satisfy** $(BC)^*$ if it preserves $(BC)^*$-squares.
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$\mathbb{P}$ satisfies fully $(BC)^*$

A monad $\mathbb{T} = (T, \mu, \eta)$ on **Set** is said to:

- satisfy (BC) if T and μ satisfy (BC).
- satisfy fully (BC) if T, μ and η satisfy (BC).

A monad $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat** is said to:

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The presheaf monad $\mathbb{P} = (P, \mathfrak{m}, \mathfrak{y})$ satisfies fully $(BC)^*$.

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$(BC)^*$ and lax idempotency

A monad $\mathbb{T} = (T, \mu, \eta)$ is lax idempotent if it satisfies $T\eta \leq \eta_T$.

Proposition (Clementino, F)

Given a monad $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat**, the following are equivalent:

(i): $\mathbb{T}$ is lax idempotent.

(ii): For each V-category (X, a), the diagram

$$\begin{array}{ccc} TX & \xrightarrow{T\eta_X} & TTX \\ {\scriptstyle \eta_{TX}}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ TTX & \xrightarrow[\mu_X]{} & TX \end{array}$$

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Submonads

Recall that a monad morphism between two monads (T,μ,η), (T',μ',η') on a category C is a natural transformation $\sigma : \mathbb{T} \to \mathbb{T}'$ such that the following diagrams commute

$$\begin{array}{ccc} 1 & \xrightarrow{\eta} & T \\ & \searrow^{\eta'} & \downarrow{\sigma} \\ & & T' \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} TT & \xrightarrow{\sigma_T} & T'T & \xrightarrow{T'\sigma} & T'T' \\ \mu\downarrow & & & & \downarrow\mu' \\ T & & \xrightarrow[\sigma]{} & & T' \end{array}$$

A **submonad** of $(P,\mathfrak{m},\mathfrak{y})$ is a monad (T,μ,η) on V-**Cat** with a monad morphism $\sigma : \mathbb{T} \to \mathbb{P}$ such that each σ_X is an embedding for every V-category X.

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Submonads of the Presheaf monad - $(BC)^*$

Given a V-functor f, we have the adjunction

$$f_* \dashv f^*.$$

It follows that we have an adjunction

$$Pf = () \cdot f^* \dashv () \cdot f_* = Qf$$

Theorem (Clementino, F)

*For a monad $\mathbb{T}$ on V-**Cat**, the following assertions are equivalent:*

(i): $\mathbb{T}$ *is a submonad of* $\mathbb{P}$.

(ii): $\mathbb{T}$ *is lax idempotent,* η *satisfies* $(BC)^*$ *and both* η_X *and* $Q\eta_X \cdot \mathfrak{y}_{TX}$ *are fully faithful for each* V*-category* (X, a).

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Submonads of the Presheaf monad - Admissible classes

We say that a Φ class of V-distributors is **admissible** if, for every V-functor $f : X \to Y$ and V-distributors $\varphi : Z \to\!\!\!\!\circ\; Y$ and $\psi : X \to\!\!\!\!\circ\; Z$ in Φ,

(i): $f^* \in \Phi$;

(ii): $\psi \cdot f^* \in \Phi$ and $f^* \cdot \varphi \in \Phi$;

(iii): $\varphi \in \Phi \Leftrightarrow \; (\forall y \in Y)\; y^* \cdot \varphi \in \Phi$;

(iv): for every V-distributor $\gamma : PX \to\!\!\!\!\circ\; E$, if the restriction of γ to ΦX belongs to Φ, then $\gamma \cdot (\mathfrak{y}_X)_* \in \Phi$.

Given a class of Φ of V-distributors, for every V-category X let

$$\Phi X = \{\varphi : X \to\!\!\!\!\circ\; E \mid \varphi \in \Phi\}$$

have the V-category structure inherited from PX.

Theorem (Clementino, Hofmann)

*For a monad $\mathbb{T}$ on V-**Cat**, the following assertions are equivalent:*

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Eilenberg-Moore Algebras

Let $\mathbb{T}$ be lax idempotent monad on V-**Cat**.

For a V-category X, the following assertions are equivalent:

(i): $\alpha\colon TX \to X$ is a $\mathbb{T}$-algebra structure on X;

(ii): there is a V-functor $\alpha\colon TX \to X$ such that $\alpha \cdot \eta_X = 1_X$;

(iii): $\alpha\colon TX \to X$ is a split epimorphism in V-**Cat**.

Given $\mathbb{T}$-algebras (X, α) and (Y, β)

$$\beta \cdot Tf \leq f \cdot \alpha,$$

for every V-functor $f\colon X \to Y$.

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Extensions

In V-**Dist**, given a V-distributor $\varphi\colon (X,a){\longrightarrow\!\!\!\!\!\!\!\circ\;\;}(Y,b)$, the functor $(\ )\cdot\varphi$ preserves suprema, and therefore it has a right adjoint $[\varphi,-]$:

$$\mathbf{Dist}(X,Z) \underset{(\ )\cdot\varphi}{\overset{[\varphi,-]}{\rightleftarrows}} \mathbf{Dist}(Y,Z)\quad \top \ .$$

For each distributor $\psi\colon X{\longrightarrow\!\!\!\!\!\!\!\circ\;\;}Z$,

$$\begin{array}{ccc} X & \overset{\psi}{\longrightarrow\!\!\!\!\!\!\!\circ\;\;} & Z \\ {\scriptstyle\varphi}\downarrow & \leq & \nearrow{\scriptstyle[\varphi,\psi]} \\ Y & & \end{array}$$

where $[\varphi,\psi]\colon Y{\longrightarrow\!\!\!\!\!\!\!\circ\;\;}Z$ is defined by $[\varphi,\psi](y,z)=\bigwedge_{x\in X}\hom(\varphi(x,y),\psi(x,z))$.

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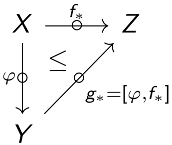
Algebras and Weighted Colimits

Given a V-functor $f\colon X \to Z$ and a distributor $\varphi\colon X \to\!\!\!\!\circ\, Y$, a φ-**colimit** of f is a V-functor $g\colon Y \to Z$ such that $g_* = [\varphi, f_*]$, if it exists.



One says then that g *represents* $[\varphi, f_*]$.

The $\mathbb{T}$-algebras for any $\mathbb{T}$ submonad of $\mathbb{P}$ can be characterized as follows:

Theorem

(i): *A map $\alpha\colon TX \to X$ is a $\mathbb{T}$-algebra structure if, and only if, for each distributor $\varphi\colon X \to\!\!\!\!\circ\, E$ in TX, $\alpha(\varphi)_* = [\varphi, (1_X)_*]$.*

(ii): *Given $\mathbb{T}$-algebras X and Y, a V-functor $f\colon X \to Y$ is a $\mathbb{T}$-algebra morphism if and only if, f preserves φ-colimits for any $\varphi \in TX$.*

Formal ball monad

The space of formal balls is an important tool in the study of (quasi)metric spaces.

Given a (quasi)metric space (X, d) its *space of formal balls* is simply the collection of all pairs (x, r), where $x \in X$ and $r \in [0, \infty[$.

This space can itself be equipped with a (quasi)metric. This construction can naturally be made into a lax idempotent monad.

Formal ball monad

The **formal ball monad** $\mathbb{B} = (B, \eta, \mu)$ is given by:

$$
\begin{aligned}
B : \textbf{Met} &\to \textbf{Met} \\
(X, a) &\mapsto BX = X \times [0, \infty[\\
(f : X \to Y) &\mapsto (Bf : BX \to BY) \\
&\qquad (x, r) \mapsto (f(x), r)
\end{aligned}
$$

where the distance in BX is given by

$$BX((x, r), (y, s)) = \hom(r, a(x, y) + s) = \max\{0, a(x, y) + s - r\}$$

$$
\begin{aligned}
\eta : X &\to BX \\
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\qquad\qquad
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Formal ball monad

The **formal ball monad** $\mathbb{B} = (B, \eta, \mu)$ is given by:

$$\begin{aligned} B : \mathbf{Met} &\to \mathbf{Met} \\ (X, a) &\mapsto BX = X \times [0, \infty[\\ (f : X \to Y) &\mapsto (Bf : BX \to BY) \\ &\qquad (x, r) \mapsto (f(x), r) \end{aligned}$$

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Extended formal ball monad

The **extended formal ball monad** $\mathbb{B}_\bullet = (B_\bullet, \eta, \mu)$ is given by:

$$\begin{aligned} B_\bullet : V\text{-}\mathbf{Cat} &\to V\text{-}\mathbf{Cat} \\ (X, a) &\mapsto B_\bullet X = X \times V \\ (f : X \to Y) &\mapsto (B_\bullet f : B_\bullet X \to B_\bullet Y) \\ &\qquad (x, r) \mapsto (f(x), r) \end{aligned}$$

where the structure in $B_\bullet X$ is given by

$$B_\bullet X((x, r), (y, s)) = \hom(r, a(x, y) \otimes s)$$

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$\mathbb{B}_\bullet$ is nearly a submonad of $\mathbb{P}$

Theorem (Clementino, F)

The natural transformation $\sigma\colon \mathbb{B}_\bullet \to \mathbb{P}$ with components defined by

$$\begin{aligned} \sigma_X : B_\bullet X &\to PX \\ (x,r) &\mapsto a(-,x)\otimes r : X \,{-\!\!\circ\!\!\!\rightarrow}\, E \end{aligned}$$

for each V-category (X,a), is a pointwise fully faithful monad morphism.

Note that $\sigma\colon \mathbb{B}_\bullet \to \mathbb{P}$ is not injective on objects; indeed, if $r=\bot$, then $\sigma_X(x,\bot)\colon X \,{-\!\!\circ\!\!\!\rightarrow}\, E$ is the distributor that is constantly $\bot$, for any $x\in X$.

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Proposition (Clementino, F)

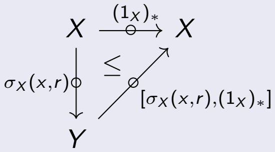
For a V-category (X, a), the following conditions are equivalent:

(i): (X, a) has a $\mathbb{B}_\bullet$-algebra structure $\alpha \colon B_\bullet X \to X$;

(ii): $(\forall x \in X)\ (\forall r \in V)\ (\exists x \oplus r \in X)\ (\forall y \in X)$

$$a(x \oplus r, y) = \hom(r, a(x, y));$$

(iii): for all $(x, r) \in B_\bullet X$, every diagram of the sort



has a (weighted) colimit.

$\mathbb{B}_\bullet$-Algebras

The V-categories X satisfying (iii) are called *tensored*. This notion was originally introduced by Borceux and Kelly for general V-categories.

Thanks to condition (ii), we also get the following characterization of tensored categories:

Corollary

A V-category (X, a) is tensored if, and only if, for every $x \in X$,

$$X \underset{x\oplus -}{\overset{a(x,-)}{\underleftarrow{\overrightarrow{\quad\top\quad}}}} V$$

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$\mathbb{B}_\circ$ is a submonad of $\mathbb{P}$ in V-$\mathbf{Cat}_{\mathrm{sep}}$ (for certain quantales)

$\mathbb{B}_\circ$ is the submonad of $\mathbb{B}_\bullet$ obtained when we only consider formal balls with radius different from $\perp$.

We (co)restricted $\mathbb{B}_\circ$ to V-$\mathbf{Cat}_{\mathrm{sep}}$ to obtain some results regarding $\mathbb{B}_\circ$-embeddings.

Unfortunately X being separated does not entail $B_\circ X$ being so. Because of this we needed also to restrict our attention to the *cancellative* quantales:

Definition

A quantale V is said to be *cancellative* if
$\forall r, s \in V, r \neq \perp : \; r = s \otimes r \;\Rightarrow\; s = k.$

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Proposition

Let V be an integral ($k = \top$) quantale. The following assertions are equivalent:

(i) $B_\circ V$ is separated;

(ii) V is cancellative;

(iii) If X is separated then $B_\circ X$ is separated.

Lastly

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Let V be a cancellative integral quantale. Then $\mathbb{B}_\circ$ is a submonad of $\mathbb{P}$ in V-$\mathbf{Cat}_{\mathrm{sep}}$.

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The characterisation of $\mathbb{B}$-algebras given in [GL19] can readily be generalised to V-**Cat** as follows:

Proposition (Clementino, F)

For a V-functor $\alpha\colon BX \to X$ the following conditions are equivalent.

- α is a $\mathbb{B}$-algebra structure.
- For every $x \in X$, $r, s \in V \setminus \{\bot\}$, $\alpha(x, k) = x$ and $\alpha(x, r \otimes s) = \alpha(\alpha(x, r), s)$.
- For every $x \in X$, $r \in V \setminus \{\bot\}$, $\alpha(x, k) = x$ and $a(x, \alpha(x, r)) \geq r$.
- For every $x \in X$, $\alpha(x, k) = x$.

Corollary (Clementino, F)

If $BX \xrightarrow{-\oplus-} X$ is a $\mathbb{B}$-algebra structure, then, for $x \in X$, $r, s \in V \setminus \{\bot\}$:

(i): $x \oplus k = x$;

(ii): $x \oplus (r \otimes s) = (x \oplus r) \oplus s$;

(iii): $a(x, x \oplus r) \geq r$.

Powerset Monad

The powerset monad $\mathcal{P} = (\mathcal{P}, \{\cdot\}, \cup)$ is given by:

$$\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$$
$$X \mapsto \mathcal{P}X = \{X \to \mathbf{2}\}$$
$$(f : X \to Y) \mapsto (\mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y)$$
$$A \mapsto f(A)$$

$$\{\cdot\} : X \to \mathcal{P}X$$
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Downset Monad

The downset monad $\mathcal{D} = (\mathcal{D}, \downarrow\{\cdot\}, \cup)$ is given by:

$$\begin{aligned} \mathcal{D} : \mathbf{Ord} &\to \mathbf{Ord} \\ X &\mapsto \mathcal{D}X = \{X^{\mathrm{op}} \to \mathbf{2}\} \\ (f : X \to Y) &\mapsto (\mathcal{D}f : \mathcal{D}X \to \mathcal{D}Y) \\ & \qquad A \mapsto \downarrow f(A) \end{aligned}$$

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Presheaf Monad

The presheaf monad $\mathbb{P} = (P, \mathfrak{y}, \mathfrak{m})$ is given by:

$$P : V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$$

$$(X, a) \mapsto PX = \{X^{\mathrm{op}} \to V\}$$

$$(f : (X, a) \to (Y, b)) \mapsto (Pf : PX \to PY)$$

$$\varphi \mapsto \varphi \cdot f^*$$

$$\mathfrak{y}_X : (X, a) \to PX$$

$$x \mapsto x^* : X^{\mathrm{op}} \to V$$

$$y \mapsto a(y, x)$$

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