On the double category of coalgebras

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Based on joint work with

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Reminder: coalgebras

Definition

For a functor $F\colon C \longrightarrow C$, one defines **coalgebra**

$$\begin{array}{ccc} FX & & FY \\ c\uparrow & & \uparrow d \\ X & & Y \end{array}$$

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For a functor $F \colon C \longrightarrow C$, one defines **coalgebra homomorphism**:

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The corresponding category of coalgebras and homomorphisms is denoted as $\mathsf{CoAlg}(F)$.

Behavioural equivalence

Definition

Let $F\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ be a functor which admits a terminal coalgebra and let (X, α) and (Y, β) be coalgebras. Then states $x \in X$ and $y \in Y$ are behaviourally equivalent whenever $!_\alpha(x) = !_\beta(y)$.

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Proposition

The following assertions are equivalent.

(i) The states $x \in X$ and $y \in Y$ are behaviourally equivalent.

(ii) There exist coalgebra homomorphisms $f\colon (X, \alpha) \longrightarrow (Z, \gamma)$ and $g\colon (Y, \beta) \longrightarrow (Z, \gamma)$ with $f(x) = g(y)$.

Bisimulations

Definition

Let (X, α) and (Y, β) be coalgebras.

1. A relation $R \subseteq X \times Y$ is a **bisimulation** whenever there is a coalgebra $\gamma\colon R \longrightarrow FR$ so that the projections $\pi_1\colon R \longrightarrow X$ and $\pi_2\colon R \longrightarrow Y$ are homomorphisms.

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Theorem

Assume that $\mathsf{F}\colon \mathrm{Set} \longrightarrow \mathrm{Set}$ preserves weak pullbacks and let (X, α) and (Y, β) be coalgebras. Then $x \in X$ and $y \in Y$ are behaviourally equivalent if and only if they are bisimilar.

Aczel, Peter and Mendler, Nax (1989). "A final coalgebra theorem". In: Category Theory and Computer Science. Ed. by David H. Pitt, David E. Rydeheard, Peter Dybjer, Andrew M. Pitts, and Axel Poigné. Springer Berlin Heidelberg, pp. 357–365.

Bisimulations via lax extensions

Theorem

Assume that $\mathsf{F}\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ preserves weak pullbacks and let (X,α) and (Y,β) be coalgebras. A relation $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ is a bisimulation if and only if

$$\begin{array}{ccc} X & \xrightarrow{\ \alpha\ } & \mathsf{F}X \\ {\scriptstyle r}\downarrow & \le & \downarrow{\scriptstyle \widehat{\mathsf{F}}r} \\ Y & \xrightarrow[\ \beta\ ]{} & \mathsf{F}Y. \end{array}$$

Rutten, Jan (1998). "Relators and Metric Bisimulations". In: Electronic Notes in Theoretical Computer Science 11, pp. 252–258.

Bisimulations via lax extensions

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Theorem

A functor $\mathsf{F}\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ preserves weak pullbacks if and only if there is an (unique) extension $\mathsf{F}\colon \mathsf{Rel} \longrightarrow \mathsf{Rel}$.

Trnková, V■ra (1977). "Relational automata in a category and their languages". In: Fundamentals of computation theory (Proc. Internat. Conf., Poznań-Kórnik, 1977). Vol. 56. Berlin: Springer, Lecture Notes in Comput. Sci., pp. 340–355.

There is more ...

- Baldan, Bonchi, Kerstan, and König study behavioural distances.

  Baldan, Paolo, Bonchi, Filippo, Kerstan, Henning, and König, Barbara (2018). "Coalgebraic Behavioral Metrics". In: Logical Methods in Computer Science 14.(3), pp. 1860–5974.

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- Worrell investigates bisimulations in the context of quantale-enriched categories.

  Worrell, James (2000). "Coinduction for recursive data types: partial orders, metric spaces and Ω-categories". In: Electronic Notes in Theoretical Computer Science. CMCS'2000, Coalgebraic Methods in Computer Science 33, pp. 337–356.

Double categories

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A double category $\mathcal{A}$ consists of objects and two types of arrows: horizontal and vertical ones, and cells in squares suggestively written as

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$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle r}\downarrow & \varepsilon & \downarrow{\scriptstyle s} \\ A & \xrightarrow[g]{} & B. \end{array}$$

We write Horiz($\mathcal{A}$) for the 2-category of horizontal arrows of $\mathcal{A}$, and with 2-cells $\varepsilon\colon g \to h$ being (from $\mathcal{A}$)

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle 1}\downarrow & \varepsilon & \downarrow{\scriptstyle 1} \\ A & \xrightarrow[g]{} & B. \end{array}$$

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Similarly, Ver($\mathcal{A}$) denotes the bicategory of vertical arrows of $\mathcal{A}$, with 2-cells $\delta\colon r \to s$ given by cells in $\mathcal{A}$ of type

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Examples: relations

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Our paradigmatic example of a double category is the double category $\mathcal{Rel}$ of sets, functions (as horizontal arrows) and relations (as vertical arrows).

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Our paradigmatic example of a double category is the double category $\mathcal{R}el$ of sets, functions (as horizontal arrows) and relations (as vertical arrows).

Example

More general, for a quantale $\mathcal{V}$, we consider the double category $\mathcal{V}$-$\mathcal{R}el$ of sets, functions (as horizontal arrows) and $\mathcal{V}$-relations (as vertical arrows).

Remark

A $\mathcal{V}$-relation from X to Y is a map $X \times Y \longrightarrow \mathcal{V}$ and it is represented by $X \longrightarrow\!\!\!\!\!\!\mapsto Y$; $\mathcal{V}$-relations can be composed via "matrix multiplication": for $r\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$ and $s\colon Y \longrightarrow\!\!\!\!\!\!\mapsto Z$,

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

Companions and conjoints

Given an horizontal arrow $f\colon A \longrightarrow B$ in a double category $\mathcal{A}$, a **companion** for f is a vertical arrow $f_*\colon A \longrightarrow\!\!\!\!\!\!\!+\;\; B$ in $\mathcal{A}$ so that

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ {\scriptstyle 1}\downarrow & \leq & \downarrow{\scriptstyle f_*} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f_*}\downarrow & \leq & \downarrow{\scriptstyle 1} \\ B & \xrightarrow[1]{} & B. \end{array}$$

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Dually, a conjoint for f is a vertical arrow $f^*\colon B \longrightarrow\!\!\!\!\!\!+\;\; A$ so that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1}\downarrow & \leq & \downarrow{\scriptstyle f^*} \\ A & \xrightarrow[1]{} & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} B & \xrightarrow{1} & B \\ {\scriptstyle f^*}\downarrow & \leq & \downarrow{\scriptstyle 1} \\ A & \xrightarrow[f]{} & B. \end{array}$$

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A & \xrightarrow[\;f\;]{} & B.
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$$

Definition

A double category $\mathcal{A}$ is a framed bicategory if every horizontal arrow has a companion and a conjoint.

Shulman, Michael (2008). "Framed bicategories and monoidal fibrations". In: Theory and Applications of Categories 20.(18), pp. 650–738.

Example

For a map $f\colon X \longrightarrow Y$, a companion of f in $\mathcal{Rel}$ is the graph $f_*\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ of f, and a conjoint is the cograph $f^*\colon Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$ of f (and similar in $\mathcal{V}$-$\mathcal{Rel}$).

The Pro-construction

Clementino, Maria Manuel, Hofmann, Dirk, and Tholen, Walter (2004). "One setting for all: metric, topology, uniformity, approach structure". In: Applied Categorical Structures 12.(2), pp. 127–154.

For a framed bicategory $\mathcal{A}$ there is the framed bicategory $\mathrm{Pro}(\mathcal{A})$ of the procompletion of $\mathrm{Ver}(\mathcal{A})$.

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- $$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ R\big\downarrow & \leq & \big\downarrow S \\ A & \xrightarrow[g]{} & B \end{array}$$ whenever

for all $s \in S$ there is $r \in R$ such that

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- $$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle R}\big\downarrow & \leq & \big\downarrow{\scriptstyle S} \\ A & \xrightarrow[g]{} & B \end{array}$$ whenever for all $s \in S$ there is $r \in R$ such that $$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle r}\big\downarrow & \leq & \big\downarrow{\scriptstyle s} \\ A & \xrightarrow[g]{} & B. \end{array}$$ in $\mathcal{A}$.
- For every horizontal arrow f in $\mathrm{Pro}(\mathcal{A})$, the set $\{f_*\}$ is a companion for f, while the set $\{f^*\}$ is a conjoint for f.

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For a framed bicategory $\mathcal{A}$, there is the framed bicategory $\mathrm{Mon}(\mathcal{A})$ of monoids, monoid homomorphisms and bimodules in $\mathcal{A}$.

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- A **monoid** in $\mathcal{A}$ consists of an object A in $\mathcal{A}$ together with a vertical arrow $a\colon A \longrightarrow\!\!\!\!\!\!\!+\;\; A$ so that $1 \leq a$ and $a \circ a \leq a$.

- A **monoid homomorphism** $f\colon (A, a) \longrightarrow (B, b)$ consists of a horizontal arrow $f\colon A \longrightarrow B \in \mathcal{A}$ so that

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle a}\big\downarrow & \leq & \big\downarrow{\scriptstyle b} \\ A & \xrightarrow[\;f\;]{} & B. \end{array}$$

- A **bimodule** $\varphi\colon (A, a) \xrightarrow{\;\circ\;} (B, b)$ consists of a vertical arrow $\varphi\colon A \longrightarrow B$ in $\mathcal{A}$ so that $\varphi \circ a \leq \varphi$ and $b \circ \varphi \leq \varphi$.

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- For every horizontal arrow $f\colon (A,a) \longrightarrow (B,b)$ in $\mathrm{Mon}(\mathcal{A})$, the vertical arrow $b \circ f_*$ in $\mathcal{A}$ is a companion for f, while the vertical arrow $f^* \circ b$ in $\mathcal{A}$ is a conjoint for f.

Examples

Example

The framed bicategory $\mathsf{Mon}(\mathcal{V}\text{-}\mathcal{R}el)$, which we denote by $\mathcal{V}\text{-}\mathcal{D}ist$, consists of $\mathcal{V}$-categories as objects, $\mathcal{V}$-functors as horizontal arrows and $\mathcal{V}$-distributors as vertical arrows.

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The framed bicategory $\mathsf{Mon}(\mathcal{V}\text{-}\mathcal{Rel})$, which we denote by $\mathcal{V}\text{-}\mathcal{Dist}$, consists of $\mathcal{V}$-categories as objects, $\mathcal{V}$-functors as horizontal arrows and $\mathcal{V}$-distributors as vertical arrows.

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The framed bicategory $\mathsf{Mon}(\mathsf{Pro}(\mathcal{Rel}))$, which we denote by $q\mathcal{Unif}$, consists of quasiuniform spaces as objects, uniformly continuous maps as horizontal arrows, and promodules as vertical arrows.

Lax double functors

A lax-double functor $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{X}$ sends

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ r \downarrow & \leq & \downarrow s \\ A & \xrightarrow[g]{} & B \end{array} \quad \text{to} \quad \begin{array}{ccc} \mathcal{F}X & \xrightarrow{\mathcal{F}f} & \mathcal{F}Y \\ \mathcal{F}r \downarrow & \leq & \downarrow \mathcal{F}s \\ \mathcal{F}A & \xrightarrow[\mathcal{F}g]{} & \mathcal{F}B \end{array}$$

and preserves horizontal composition and identities strictly and vertical composition and identities laxly.

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A lax-framed functor is a lax-double functor between framed bicategories.

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Theorem

A lax-framed functor $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{X}$ corresponds precisely to a pair $(\mathrm{F}, \widehat{\mathrm{F}})$, where $\mathrm{F}\colon \mathrm{Horiz}(\mathcal{A}) \longrightarrow \mathrm{Horiz}(\mathcal{X})$ is a 2-functor and $\widehat{\mathrm{F}}\colon \mathrm{Ver}(\mathcal{A}) \longrightarrow \mathrm{Ver}(\mathcal{X})$ is a lax functor, such that for every $f\colon X \longrightarrow Y \in \mathcal{A}$,

$$\mathrm{F}(f)_* \leq \widehat{\mathrm{F}}(f_*) \text{ and } \mathrm{F}(f)^* \leq \widehat{\mathrm{F}}(f^*).$$

Coalgebras

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- the vertical arrows in $\mathsf{CoAlg}(\mathcal{F})$ between objects (A, α) and (B, β) are $\mathcal{F}$-simulations, that is, vertical arrows $s\colon X \nrightarrow Y$ in $\mathcal{A}$ so that

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & \mathsf{F}A \\ {\scriptstyle s}\downarrow & \leq & \downarrow{\scriptstyle \widehat{\mathsf{F}}s} \\ B & \xrightarrow[\beta]{} & \mathsf{F}B. \end{array}$$

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Theorem

$\mathsf{CoAlg}(\mathcal{F})$ is a framed bicategory, for a lax-framed functor $\mathcal{F}$.

Remark

For a coalgebra homomorphism $f\colon (A, \alpha) \longrightarrow (B, \beta)$, the companion f_* and the conjoint f^* of $f\colon A \longrightarrow B$ in $\mathcal{A}$ are $\widehat{\mathsf{F}}$-simulations.

Similarity

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Definition

Let $\mathcal{F} \colon \mathcal{A} \longrightarrow \mathcal{A}$ be a lax-framed functor where $\mathcal{A}$ is locally complete. Let (A, α) and (B, β) be $\mathcal{F}$-coalgebras.

The $\mathcal{F}$-similarity from (A, α) to (B, β) is the greatest $\mathcal{F}$-simulation from (A, α) to (B, β), and we denote it by $\top_{\alpha,\beta}$, or by $\top_{\alpha}$, if $\alpha = \beta$.

Similarity

Definition

We call a framed bicategory $\mathcal{A}$ locally complete whenever the ordered category $\mathrm{Ver}(\mathcal{A})$ has complete hom-sets.

Remark

The previous constructions preserve local completeness.

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Theorem

Let $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{A}$ be a lax-framed functor where $\mathcal{A}$ is locally complete. For every pair of horizontal arrows

$$(A, \alpha) \xrightarrow{f} (C, \gamma) \qquad\qquad (B, \beta) \xrightarrow{g} (D, \delta)$$

in $\mathrm{CoAlg}\,\mathcal{F}$, $\top_{\alpha,\beta} = g^* \circ \top_{\gamma,\delta} \circ f_*$.

Behavioural distance

Remark

For every lax-framed functor $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{A}$ on a locally complete framed bicategory, the forgetful functor

$$\mathsf{Horiz}(\mathsf{CoAlg}\,\mathsf{Mon}(\mathcal{F})) \longrightarrow \mathsf{Horiz}(\mathsf{CoAlg}\,\mathcal{F})$$

is topological and therefore has a right adjoint

$$\mathsf{gfp}\colon\ \mathsf{Horiz}(\mathsf{CoAlg}\,\mathcal{F}) \longrightarrow \mathsf{Horiz}(\mathsf{CoAlg}\,\mathsf{Mon}(\mathcal{F})).$$

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Definition

Let $\mathcal{F}$ be a lax-framed endofunctor on a locally complete framed bicategory. The $\mathcal{F}$-behavioural distance bd_α on an $\mathcal{F}$-coalgebra (A, α) is the monoid structure on A given by $\mathsf{gfp}(A, \alpha)$.

Behavioural distance = similarity

Theorem
Let $\mathcal{F}$ be a lax-framed endofunctor on a locally complete framed bicategory. Then, $\mathcal{F}$-similarity and $\mathcal{F}$-behavioural distance coincide on every $\mathcal{F}$-coalgebra.

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Corollary

$\mathcal{F}$-behavioural distance is compatible with coalgebra homomorphisms.

Worrell (2000)

Worrell, James (2000). "Coinduction for recursive data types: partial orders, metric spaces and Ω-categories". In: Electronic Notes in Theoretical Computer Science. CMCS'2000, Coalgebraic Methods in Computer Science 33, pp. 337–356.

Worrell considers a locally monotone functor $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$, where $\mathcal{V}$ is a quantale.

Theorem

Assume that $\mathsf{F}\colon \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ preserves initial $\mathcal{V}$-functors and admits a final coalgebra (Z, c, γ). For every coalgebra (X, a, α) and all $x, y \in X$,

$$c(\mathrm{beh}_\alpha(x), \mathrm{beh}_\alpha(y)) = \bigvee \{\varphi(x, y) \mid \varphi\colon (X, a) \relbar\joinrel\mapstochar\joinrel\rightarrow (X, a) \text{ bisimulation}\}.$$

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Theorem

F extends (in a canonical way) to a normal lax functor $\widehat{\mathsf{F}}\colon \mathcal{V}\text{-}\mathsf{Dist} \longrightarrow \mathcal{V}\text{-}\mathsf{Dist}$ if and only if F preserves initial $\mathcal{V}$-functors.

From our perspective

Corollary

Consider a normal lax-double functor $\mathcal{F}\colon \mathcal{V}\text{-}\mathcal{D}ist \longrightarrow \mathcal{V}\text{-}\mathcal{D}ist$ which admits a admits a terminal coalgebra (Z, c, γ). For every coalgebra (X, a, α) and all $x, y \in X$,

$$\mathrm{bd}_\gamma(\mathrm{beh}_\alpha(x), \mathrm{beh}_\alpha(y)) = \top_\alpha(x, y).$$

Remark

Above we apply (on the horizontal categories)

$$\mathrm{CoAlg}(\mathcal{F} \text{ on } \mathcal{V}\text{-}\mathcal{D}ist) \longrightarrow \mathrm{CoAlg}(\mathrm{Mon}(\mathcal{F}) \text{ on } \mathrm{Mon}(\mathcal{V}\text{-}\mathcal{D}ist))$$

which, being right adjoint, preserves terminal coalgebras.

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which, being right adjoint, preserves terminal coalgebras.

Lemma

Let $\mathcal{A}$ be a framed bicategory and let $\mathcal{F}\colon \mathsf{Mon}(\mathcal{A}) \longrightarrow \mathsf{Mon}(\mathcal{A})$ be a normal lax-double functor. If (C, c, γ) is a terminal $\mathcal{F}$-coalgebra, then (C, c, c, γ) is a terminal $\mathsf{Mon}(\mathcal{F})$-coalgebra.