

# Universal indexed categories

Matthew Di Meglio



THE UNIVERSITY  
*of* EDINBURGH

TACL2022

① Introduction to indexed categories

② Universality of self indexing

③ Indexed monoidal categories

Families of objects and morphisms are  
*ubiquitous* and *fundamental* in category theory

Families of objects and morphisms are *ubiquitous* and *fundamental* in category theory

Concept

Family

category  $\underline{C}$

$(\underline{C}(X, Y))_{X, Y \in \text{ob } \underline{C}}$

Families of objects and morphisms are *ubiquitous* and *fundamental* in category theory

Concept

Family

category  $\underline{C}$

$(\underline{C}(X, Y))_{X, Y \in \text{ob } \underline{C}}$

functor  $\underline{C} \xrightarrow{F} \underline{D}$

$(\underline{C}(X, Y) \xrightarrow{F_{X, Y}} \underline{D}(FX, FY))_{X, Y \in \text{ob } \underline{C}}$

Families of objects and morphisms are *ubiquitous* and *fundamental* in category theory

Concept

Family

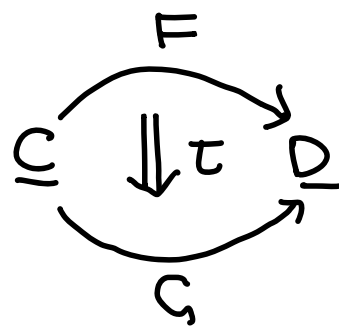
category  $\underline{C}$

$(\underline{C}(X, Y))_{X, Y \in \text{ob } \underline{C}}$

functor  $\underline{C} \xrightarrow{F} \underline{D}$

$(\underline{C}(X, Y) \xrightarrow{F_{X, Y}} \underline{D}(FX, FY))_{X, Y \in \text{ob } \underline{C}}$

natural transformation



$(FX \xrightarrow{\tau_x} GX)_{X \in \text{ob } \underline{C}}$

## Indexed categories

2

$\underline{S}$  = category with a terminal object  $1$

An  $\underline{S}$ -indexing  $\mathbb{C}$  of a category  $\underline{C}$  consists of

## Indexed categories

2

$\underline{S}$  = category with a terminal object 1

An  $\underline{S}$ -indexing  $\mathbb{C}$  of a category  $\underline{C}$  consists of

- for all  $J$  in  $\underline{S}$ , a category  $\mathbb{C}^J$  with  $\mathbb{C}^1 \cong \underline{C}$   
(objects and morphisms of  $\mathbb{C}^J$  are  $J$ -indexed families of those of  $\underline{C}$ )



# Indexed categories

2

$\underline{S}$  = category with a terminal object 1

An  $\underline{S}$ -indexing  $\mathbb{C}$  of a category  $\underline{C}$  consists of

- for all  $J$  in  $\underline{S}$ , a category  $\mathbb{C}^J$  with  $\mathbb{C}^1 \cong \underline{C}$   
(objects and morphisms of  $\mathbb{C}^J$  are  $J$ -indexed families of those of  $\underline{C}$ )
- for all  $r: J \rightarrow K$  in  $\underline{S}$ , a functor  $\Delta_r: \mathbb{C}^K \rightarrow \mathbb{C}^J$   
( $\Delta_r$  reindexes  $K$ -indexed families along  $r$ )

# Indexed categories

2

$\underline{S}$  = category with a terminal object  $1$

An  $\underline{S}$ -indexing  $\mathbb{C}$  of a category  $\underline{C}$  consists of

- for all  $J$  in  $\underline{S}$ , a category  $\mathbb{C}^J$  with  $\mathbb{C}^1 \cong \underline{C}$   
(objects and morphisms of  $\mathbb{C}^J$  are  $J$ -indexed families of those of  $\underline{C}$ )
- for all  $r: J \rightarrow K$  in  $\underline{S}$ , a functor  $\Delta_r: \mathbb{C}^K \rightarrow \mathbb{C}^J$   
( $\Delta_r$  reindexes  $K$ -indexed families along  $r$ )

such that  $\Delta_{rs} \cong \Delta_s \Delta_r$  and  $\Delta_{id_J} \cong id_{\mathbb{C}^J}$  coherently.  
(i.e. the data forms a pseudofunctor  $\underline{S}^{op} \rightarrow \underline{Cat}$ )

# Indexed categories

2

$\underline{S}$  = category with a terminal object 1

An  $\underline{S}$ -indexing  $\underline{C}$  of a category  $\underline{C}$  consists of

- for all  $J$  in  $\underline{S}$ , a category  $\underline{C}^J$  with  $\underline{C}^1 \cong \underline{C}$   
(objects and morphisms of  $\underline{C}^J$  are  $J$ -indexed families of those of  $\underline{C}$ )
- for all  $r: J \rightarrow K$  in  $\underline{S}$ , a functor  $\Delta_r: \underline{C}^K \rightarrow \underline{C}^J$   
( $\Delta_r$  reindexes  $K$ -indexed families along  $r$ )

such that  $\Delta_{rs} \cong \Delta_s \Delta_r$  and  $\Delta_{id_J} \cong id_{\underline{C}^J}$  coherently.  
(i.e. the data forms a pseudofunctor  $\underline{S}^{op} \rightarrow \underline{Cat}$ )

Example (set indexing)

$$\underline{S} = \underline{Set}$$

$$\underline{C}^J = \prod_{j \in J} \underline{C}$$

$$\Delta_r(Y_k)_{k \in K} = (Y_{r_j})_{j \in J}$$

# Indexed categories

2

$\underline{S}$  = category with a terminal object  $1$

An  $\underline{S}$ -indexing  $\mathbb{C}$  of a category  $\underline{C}$  consists of

- for all  $J$  in  $\underline{S}$ , a category  $\mathbb{C}^J$  with  $\mathbb{C}^1 \cong \underline{C}$   
(objects and morphisms of  $\mathbb{C}^J$  are  $J$ -indexed families of those of  $\underline{C}$ )
- for all  $r: J \rightarrow K$  in  $\underline{S}$ , a functor  $\Delta_r: \mathbb{C}^K \rightarrow \mathbb{C}^J$   
( $\Delta_r$  reindexes  $K$ -indexed families along  $r$ )

such that  $\Delta_{rs} \cong \Delta_s \Delta_r$  and  $\Delta_{id_J} \cong id_{\mathbb{C}^J}$  coherently.  
(i.e. the data forms a pseudofunctor  $\underline{S}^{op} \rightarrow \underline{Cat}$ )

Example (set indexing)

$$\underline{S} = \underline{Set}$$

$$\mathbb{C}^J = \prod_{j \in J} \underline{C}$$

$$\Delta_r(\gamma_k)_{k \in K} = (\gamma_{rj})_{j \in J}$$

Example (self indexing)

$$\underline{S} = \text{finitely complete } \underline{C}$$

$$\mathbb{C}^J = \underline{C}/J$$

$$\Delta_r(\gamma \xrightarrow{y} J) = \text{chosen pullback of } y \text{ along } r$$

## Indexed sums

3

An indexed category has **indexed sums** if each  $\Delta_r$  has a left adjoint  $\Sigma_r$  that is compatible with reindexing.

## Indexed sums

3

An indexed category has **indexed sums** if each  $\Delta_r$  has a left adjoint  $\Sigma_r$  that is compatible with reindexing.

### Proposition

A category has small sums if and only if its set indexing has indexed sums.

$$\Sigma_r (X_j)_{j \in J} = \left( \sum_{j \in r^{-1}\{k\}} X_j \right)_{k \in K}$$

## Indexed sums

3

An indexed category has **indexed sums** if each  $\Delta_r$  has a left adjoint  $\Sigma_r$  that is compatible with reindexing.

### Proposition

A category has small sums if and only if its set indexing has indexed sums.

$$\Sigma_r (X_j)_{j \in J} = \left( \sum_{j \in r^{-1}\{k\}} X_j \right)_{k \in K}$$

### Proposition

The self indexing of a finitely complete category has indexed sums.

$$\Sigma_r (X \xrightarrow{x} J) = (X \xrightarrow{rx} K)$$

## Extensivity

4

An indexed category  $\underline{\mathcal{C}} \xrightarrow{\mathcal{C}} \underline{\mathcal{C}at}$  is **extensive** if

- it has indexed sums, and
- for all  $r: J \rightarrow K$  in  $\underline{\mathcal{C}}$  and all  $X$  in  $\mathcal{C}^J$ , the functor

$$\Sigma_r : \mathcal{C}^J / X \longrightarrow \mathcal{C}^K / \Sigma_r X$$

is an equivalence of categories.



## Extensivity

4

An indexed category  $\underline{\mathcal{C}} \xrightarrow{\mathcal{C}} \underline{\text{Cat}}$  is **extensive** if

- it has indexed sums, and
- for all  $r: J \rightarrow K$  in  $\underline{\mathcal{C}}$  and all  $X$  in  $\mathcal{C}^J$ , the functor

$$\Sigma_r: \mathcal{C}^J / X \longrightarrow \mathcal{C}^K / \Sigma_r X$$

is an equivalence of categories.

### Proposition

A category is extensive if and only if its set indexing is too.

## Extensivity

4

An indexed category  $\underline{\mathcal{C}} \xrightarrow{\mathcal{C}} \underline{\text{Cat}}$  is **extensive** if

- it has indexed sums, and
- for all  $r: J \rightarrow K$  in  $\underline{\mathcal{C}}$  and all  $X$  in  $\mathcal{C}^J$ , the functor

$$\Sigma_r: \mathcal{C}^J / X \longrightarrow \mathcal{C}^K / \Sigma_r X$$

is an equivalence of categories.

### Proposition

A category is extensive if and only if its set indexing is too.

### Proposition

The self indexing of a finitely complete category is extensive.

① Introduction to indexed categories

② Universality of self indexing

③ Indexed monoidal categories

# Universality of self indexing

5

ExtIndLex

Extensive finitely complete indexed categories and  
finitely continuous indexed functors that preserve indexed coproducts

# Universality of self indexing

5

$$(\underline{S}^{\text{op}} \xrightarrow{\mathbb{C}} \underline{\text{Lex}}) \dashv \dashv \rightarrow \mathbb{C}^1$$

underlying category

ExtIndLex

Lex

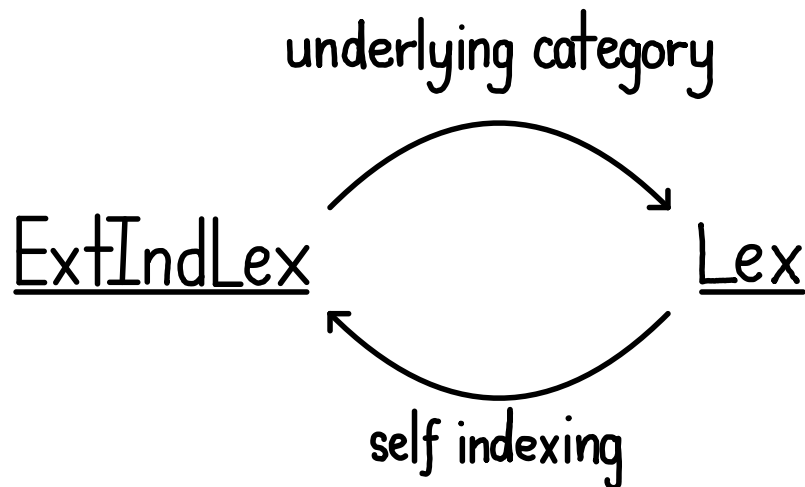


Extensive finitely complete indexed categories and finitely continuous indexed functors that preserve indexed coproducts

# Universality of self indexing

5

$$(\underline{S}^{\text{op}} \xrightarrow{\mathbb{C}} \underline{\text{Lex}}) \dashv \dashv \mathbb{C}^1$$

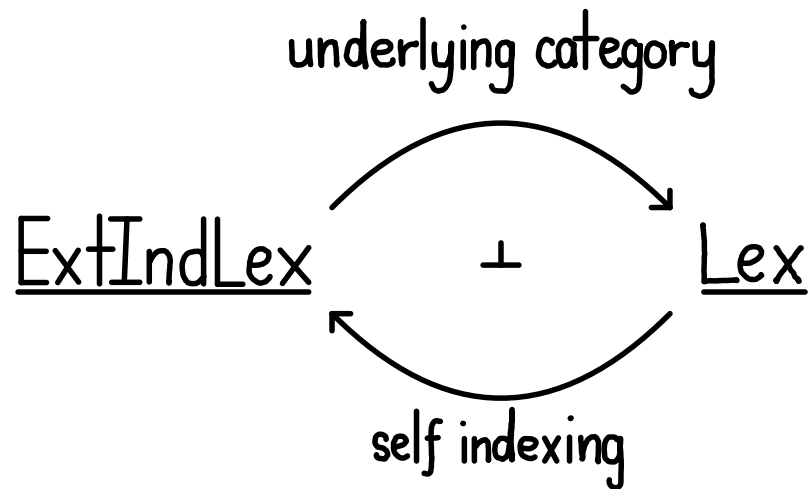


$$(\underline{D}^{\text{op}} \xrightarrow{\underline{D}/(\varepsilon)} \underline{\text{Lex}}) \dashv \dashv \underline{D}$$

Extensive finitely complete indexed categories and finitely continuous indexed functors that preserve indexed coproducts

# Universality of self indexing

$$(\underline{S}^{\text{op}} \xrightarrow{\underline{C}} \underline{\text{Lex}}) \dashv \dashv \rightarrow \underline{C}^1$$

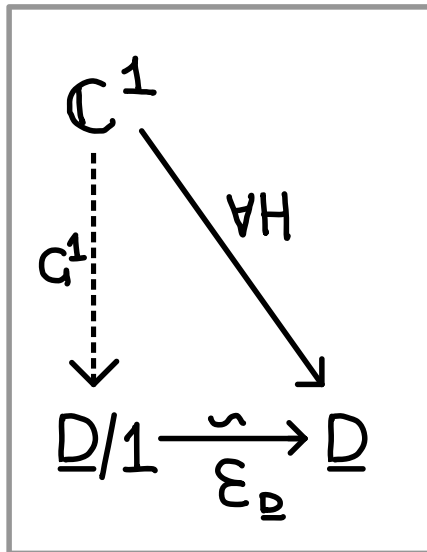
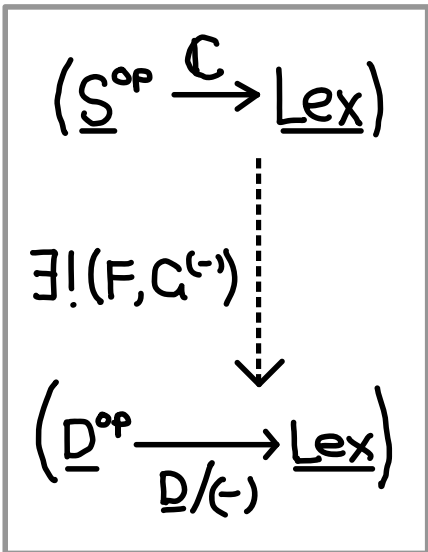
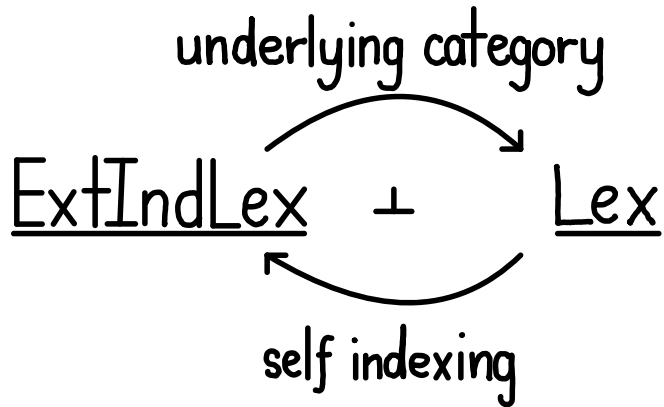


$$(\underline{D}^{\text{op}} \xrightarrow{\underline{D}/(\varepsilon)} \underline{\text{Lex}}) \dashv \dashv \leftarrow \underline{D}$$



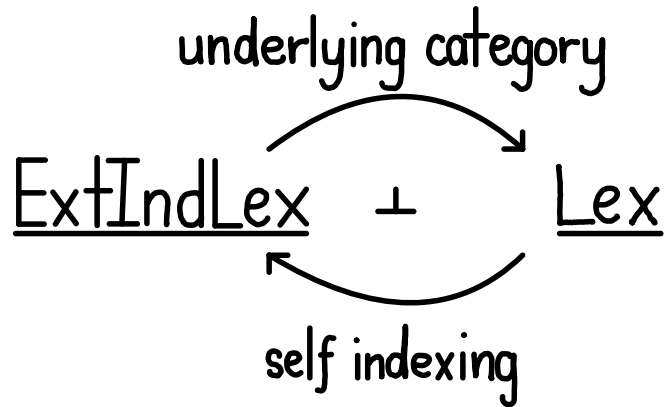
Extensive finitely complete indexed categories and finitely continuous indexed functors that preserve indexed coproducts

# Universality of self indexing

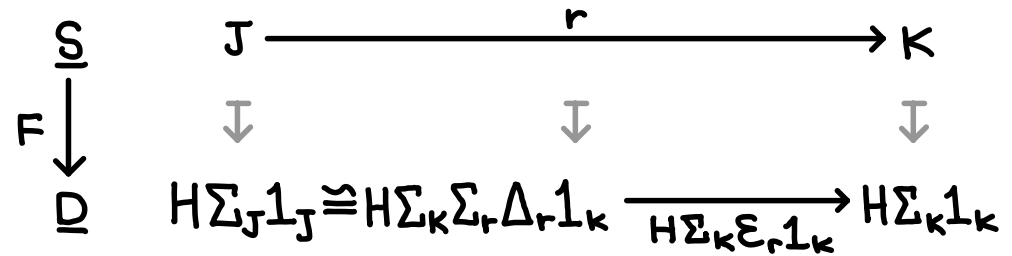




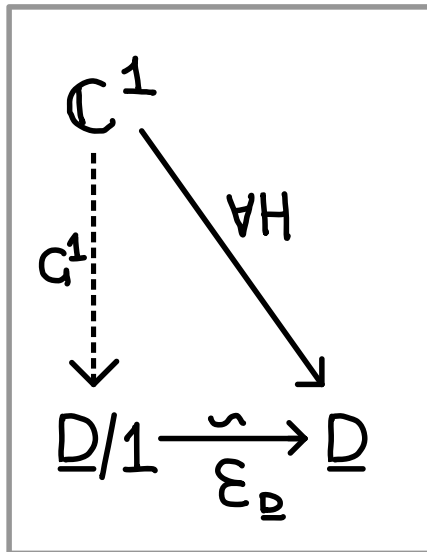
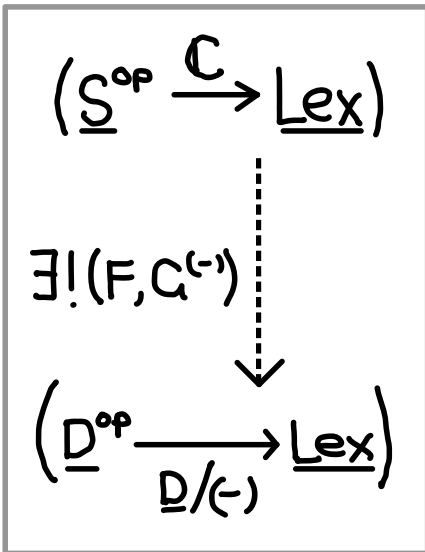
# Universality of self indexing



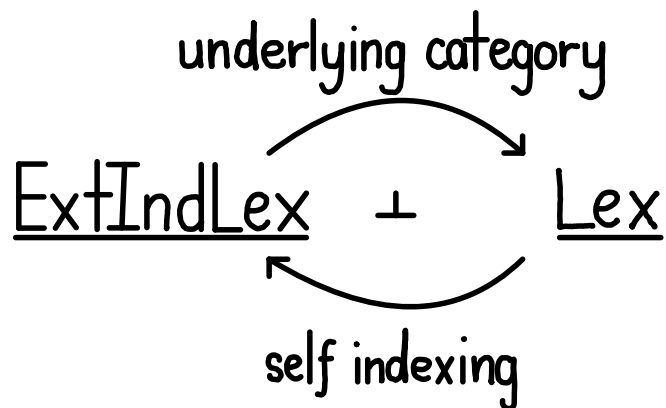
## Existence



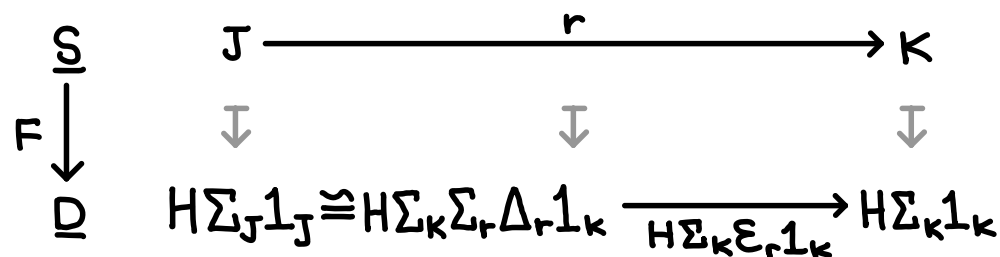
$$C^J = (C^J \cong C^J/1_J \xrightarrow{\Sigma_J} C^1/\Sigma_J 1_J \xrightarrow{H} \underline{D}/FJ)$$



# Universality of self indexing



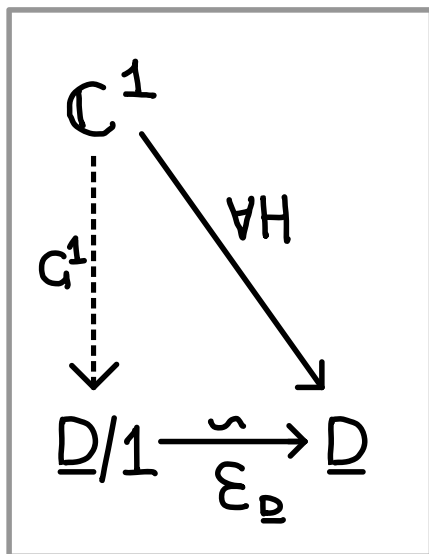
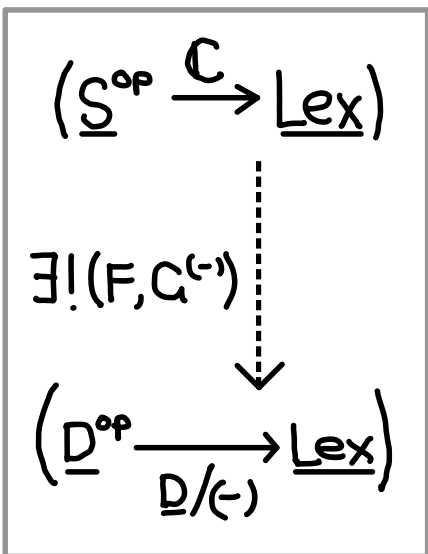
## Existence



$$C^J = (C^J \cong C^J/1_J \xrightarrow{\Sigma_J} C^1/\Sigma_J 1_J \xrightarrow{H} D/FJ)$$

## Remarks

- Easier to prove strict functoriality and uniqueness using fibrations
- Need extensivity to show that  $G$  is compatible with  $\Delta$
- Construction same as Moens (1982)



① Introduction to indexed categories

② Universality of self indexing

③ Indexed monoidal categories

# Indexed monoidal categories

7

A symmetric monoidal  $\underline{S}$ -indexing of a symmetric monoidal category  $\mathcal{V}$  is a pseudofunctor

$$\mathbb{V}: \underline{S}^{\text{op}} \rightarrow \underline{\text{SymMonCat}}$$

strong symmetric  
monoidal functors

where  $\underline{S}$  is cartesian monoidal and  $\mathbb{V}^1 \cong \mathcal{V}$ .

# Indexed monoidal categories

7

A symmetric monoidal  $\underline{S}$ -indexing of a symmetric monoidal category  $\mathcal{V}$  is a pseudofunctor

$$\mathbb{V}: \underline{S}^{\text{op}} \rightarrow \underline{\text{SymMonCat}}$$

strong symmetric  
monoidal functors

where  $\underline{S}$  is cartesian monoidal and  $\mathbb{V}^1 \cong \mathcal{V}$ .

Example (set indexing)

$$\underline{S} = \underline{\text{Set}} \quad \mathbb{V}^J = \prod_{j \in J} \mathcal{V} \quad (X_j)_{j \in J} \otimes_J (Y_j)_{j \in J} = (X_j \otimes Y_j)_{j \in J}$$

# Indexed monoidal categories

7

A symmetric monoidal  $\underline{S}$ -indexing of a symmetric monoidal category  $\mathcal{V}$  is a pseudofunctor

$$\mathbb{V}: \underline{S}^{\text{op}} \rightarrow \underline{\text{SymMonCat}}$$

strong symmetric monoidal functors

where  $\underline{S}$  is cartesian monoidal and  $\mathbb{V}^1 \cong \mathcal{V}$ .

Example (set indexing)

$$\underline{S} = \underline{\text{Set}} \quad \mathbb{V}^J = \prod_{j \in J} \mathcal{V} \quad (X_j)_{j \in J} \otimes_J (Y_j)_{j \in J} = (X_j \otimes Y_j)_{j \in J}$$

Non-example (self indexing)

$$\underline{S} = \mathcal{V} \quad \mathbb{V}^J = \mathcal{V}/J \quad (X \xrightarrow{x} J) \otimes_J (Y \xrightarrow{y} J) = ?$$

# Comonoids

8

A (co)commutative comonoid  $J$  is an object  $J$  equipped with morphisms

$$d_J : J \rightarrow J \otimes J$$

comultiplication

and

$$e_J : J \rightarrow I$$

counit

subject to coassociativity, counitality and cocommutativity laws.

# Comonoids

8

A (co)commutative comonoid  $J$  is an object  $J$  equipped with morphisms

$$\begin{array}{ccc} d_J : J \rightarrow J \otimes J & \text{and} & e_J : J \rightarrow I \\ \text{comultiplication} & & \text{counit} \end{array}$$

subject to coassociativity, counitality and cocommutativity laws.

Comon $_{\mathcal{V}}$  = category of comonoids in  $\mathcal{V}$



# Comonoids

8

A (co)commutative comonoid  $J$  is an object  $J$  equipped with morphisms

$$\begin{array}{ccc} d_J : J \rightarrow J \otimes J & \text{and} & e_J : J \rightarrow I \\ \text{comultiplication} & & \text{counit} \end{array}$$

subject to coassociativity, counitality and cocommutativity laws.

Comon <sub>$\mathcal{V}$</sub>  = category of comonoids in  $\mathcal{V}$

## Proposition

If  $\mathcal{V}$  is cartesian monoidal, then Comon <sub>$\mathcal{V}$</sub>   $\cong$   $\mathcal{V}$

# Comonoids

8

A (co)commutative comonoid  $J$  is an object  $J$  equipped with morphisms

$$\begin{array}{ccc} d_J : J \rightarrow J \otimes J & \text{and} & e_J : J \rightarrow I \\ \text{comultiplication} & & \text{counit} \end{array}$$

subject to coassociativity, counitality and cocommutativity laws.

Comon $_{\mathcal{V}}$  = category of comonoids in  $\mathcal{V}$

## Proposition

If  $\mathcal{V}$  is cartesian monoidal, then Comon $_{\mathcal{V}} \cong \mathcal{V}$

## Proposition (Fox 1975)

Comon $_{\mathcal{V}}$  is the cofree cartesian monoidal category on  $\mathcal{V}$

# Comodules

9

A  $J$ -comodule  $(X, \alpha)$  is an object  $X$  equipped with a morphism

$$\alpha: X \longrightarrow J \otimes X$$

coaction

that preserves comultiplication and counits.

# Comodules

9

A  $J$ -comodule  $(X, \alpha)$  is an object  $X$  equipped with a morphism

$$\alpha: X \longrightarrow J \otimes X$$

coaction

that preserves comultiplication and counits.

Comod <sub>$\mathcal{V}$</sub> ( $J$ ) = category of  $J$ -comodules in  $\mathcal{V}$

# Comodules

9

A  $J$ -comodule  $(X, \alpha)$  is an object  $X$  equipped with a morphism

$$\alpha: X \longrightarrow J \otimes X$$

coaction

that preserves comultiplication and counits.

Comod <sub>$\mathcal{V}$</sub> ( $J$ ) = category of  $J$ -comodules in  $\mathcal{V}$

## Proposition

If  $\mathcal{V}$  is cartesian monoidal, then Comod <sub>$\mathcal{V}$</sub> ( $J$ )  $\cong$   $\mathcal{V}/J$

# Comonoid indexing

10

The comonoid indexing of a symmetric monoidal category  $\mathcal{V}$  with nice equalisers is the indexed category

$$\underline{\text{Comod}}_{\mathcal{V}}(-) : \underline{\text{Comon}}_{\mathcal{V}}^{\text{op}} \rightarrow \underline{\text{SymMonCat}}$$

# Comonoid indexing

The comonoid indexing of a symmetric monoidal category  $\mathcal{V}$  with nice equalisers is the indexed category

$$\underline{\text{Comod}}_{\mathcal{V}}(-) : \underline{\text{Comon}}_{\mathcal{V}}^{\text{op}} \rightarrow \underline{\text{SymMonCat}}$$

## Proposition

If  $\mathcal{V}$  is cartesian monoidal, then it is finitely complete and its comonoid indexing is isomorphic with its self indexing.

## Monoidal extensivity

A symmetric monoidal category  $\mathcal{V}$  is (infinitary) monoidal extensivity if it has small coproducts and the functor

$$\Sigma : \prod_{j \in J} \underline{\text{Comod}}_{\mathcal{V}}(A_j) \longrightarrow \underline{\text{Comod}}_{\mathcal{V}}\left(\sum_{j \in J} A_j\right)$$

is always an equivalence of categories.

Example (Grunenfelder-Paré 1987)  $\underline{\text{Vect}}_{\mathbb{k}}$



## Monoidal extensivity

A symmetric monoidal category  $\mathcal{V}$  is (infinitary) **monoidal extensive** if it has small coproducts and the functor

$$\Sigma : \prod_{j \in J} \underline{\text{Comod}}_{\mathcal{V}}(A_j) \longrightarrow \underline{\text{Comod}}_{\mathcal{V}}\left(\sum_{j \in J} A_j\right)$$

is always an equivalence of categories.

Example (Grunenfelder-Paré 1987)  $\underline{\text{Vect}}_{\mathbb{K}}$

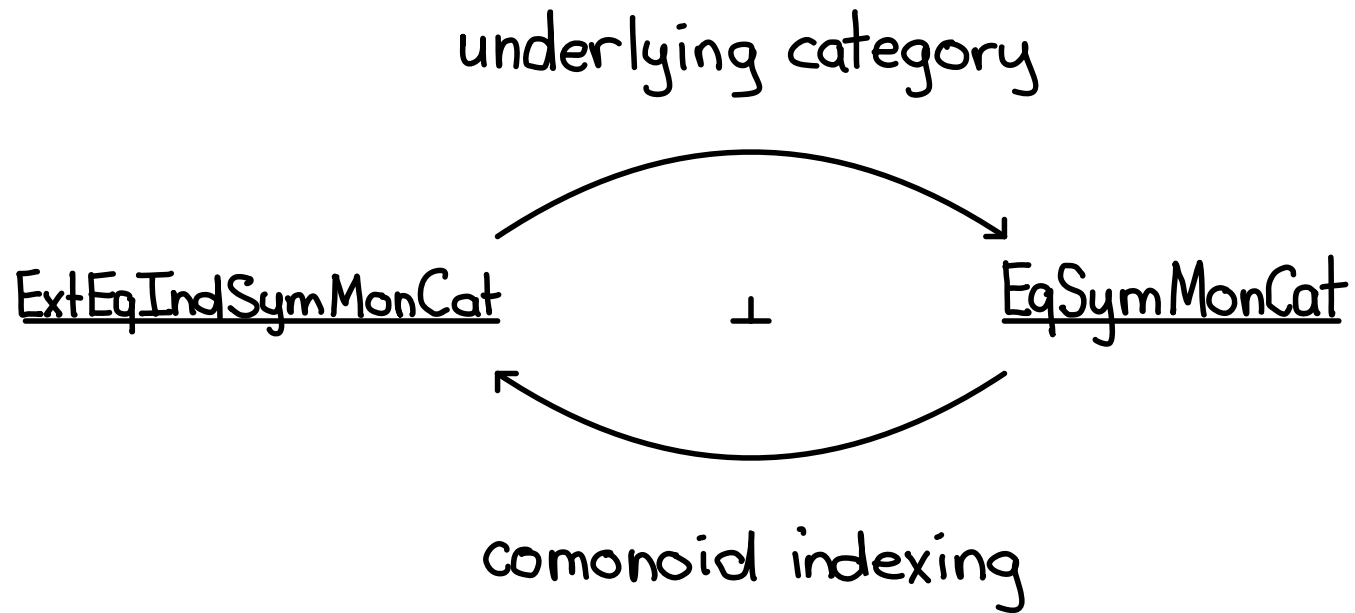
An indexed symmetric monoidal category  $\mathbb{V} : \underline{\mathbb{S}}^{\text{op}} \rightarrow \underline{\text{SymMonCat}}$  is **monoidal extensive** if it has indexed coproducts and the functor

$$\Sigma_r : \text{Comod}_{\mathbb{V}J}(A) \longrightarrow \text{Comod}_{\mathbb{V}K}(\Sigma_r A)$$

is always an equivalence of categories.

# Universality of comonoid indexing \*

12



\* work in progress

## Conclusion

- Comonoid indexing of nice symmetric monoidal categories generalises self indexing of finitely complete categories
- Taking the self indexing/comonoid indexing is right adjoint to taking the underlying category
- There is a monoidal generalisation of extensivity

## Conclusion

- Comonoid indexing of nice symmetric monoidal categories generalises self indexing of finitely complete categories
- Taking the self indexing/comonoid indexing is right adjoint to taking the underlying category
- There is a monoidal generalisation of extensivity

## Future work

- Finish checking details for monoidal case
- Check monoidal version of Moens' theorem
- Investigate links to linear dependent types
- Work out link between categories internal to and enriched in a monoidal category via monoidal extensivity