Universal indexed categories

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THE UNIVERSITY of EDINBURGH

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1. Introduction to indexed categories
2. Universality of self indexing
3. Indexed monoidal categories

Families of objects and morphisms are *ubiquitous* and *fundamental* in category theory

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functor $\underline{C} \xrightarrow{F} \underline{D}$	$\big(\underline{C}(X,Y) \xrightarrow{F_{X,Y}} \underline{D}(FX,FY)\big)_{X,Y \in \mathrm{ob}\,\underline{C}}$

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natural transformation $\begin{tikzcd} \underline{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\tau"] & \underline{D} \end{tikzcd}$	$\left(FX \xrightarrow{\tau_X} GX\right)_{X \in \mathrm{ob}\,\underline{C}}$

Indexed categories

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Example (set indexing)

$$\underline{S} = \underline{Set}$$

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Example (self indexing)

$\underline{S}$ = finitely complete $\underline{C}$

$\mathbb{C}^J = \underline{C}/J$

$\Delta_r(Y \xrightarrow{y} J)$ = chosen pullback of y along r

Indexed sums

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Proposition

A category has small sums if and only if its set indexing has indexed sums.

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Proposition

The self indexing of a finitely complete category has indexed sums.

$$\Sigma_r (X \xrightarrow{x} J) = (X \xrightarrow{rx} K)$$

Extensivity

An indexed category $\underline{S}^{op} \xrightarrow{\mathbb{C}} \underline{Cat}$ is extensive if

- it has indexed sums, and
- for all $r : J \longrightarrow K$ in $\underline{S}$ and all X in C^J, the functor

$$\Sigma_r : C^J/X \longrightarrow C^K/\Sigma_r X$$

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Proposition

The self indexing of a finitely complete category is extensive.

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Universality of self indexing

ExtIndLex

Extensive finitely complete indexed categories and finitely continuous indexed functors that preserve indexed coproducts

Universality of self indexing

$$(\underline{S}^{op} \xrightarrow{\mathbb{C}} \underline{\mathrm{Lex}}) \longmapsto \mathbb{C}^1$$

$$\underline{\mathrm{ExtIndLex}} \xrightarrow{\text{underlying category}} \underline{\mathrm{Lex}}$$

Extensive finitely complete indexed categories and finitely continuous indexed functors that preserve indexed coproducts

Universality of self indexing

$$(\underline{S}^{op} \xrightarrow{C} \underline{Lex}) \longmapsto C^1$$

$$\begin{tikzcd}
\underline{ExtIndLex} \arrow[rr, bend left, "\text{underlying category}"] & & \underline{Lex} \arrow[ll, bend left, "\text{self indexing}"]
\end{tikzcd}$$

$$(\underline{D}^{op} \xrightarrow{\underline{D}/(-)} \underline{Lex}) \longleftarrow\!\shortmid \underline{D}$$

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\left(\underline{S}^{op} \arrow[r, "\mathbb{C}"] & \underline{\mathrm{Lex}}\right) \arrow[d, dashed, "\exists!(F, G^{(-)})"'] \\
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\begin{tikzcd}
\mathbb{C}^1 \arrow[d, dashed, "G^1"'] \arrow[dr, "\forall H"] & \\
\underline{D}/1 \arrow[r, "\simeq", "\varepsilon_{\underline{D}}"'] & \underline{D}
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Existence

$$
\begin{tikzcd}
\underline{S} \arrow[d, "F"'] & J \arrow[rr, "r"] \arrow[d, mapsto] & & K \arrow[d, mapsto] \\
\underline{D} & H\Sigma_J 1_J \cong H\Sigma_K \Sigma_r \Delta_r 1_K \arrow[rr, "H\Sigma_K \varepsilon_r 1_K"'] & & H\Sigma_K 1_K
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$$
G^{J} = \left(\mathbb{C}^{J} \cong \mathbb{C}^{J}/1_J \xrightarrow{\Sigma_J} \mathbb{C}^{1}/\Sigma_J 1_J \xrightarrow{H} \underline{D}/FJ\right)
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$$G^J = \left(\mathbb{C}^J \cong \mathbb{C}^J/1_J \xrightarrow{\Sigma_J} \mathbb{C}^1/\Sigma_J 1_J \xrightarrow{H} \underline{D}/FJ\right)$$

Remarks

- Easier to prove strict functoriality and uniqueness using fibrations
- Need extensivity to show that G is compatible with Δ
- Construction same as Moens (1982)

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Indexed monoidal categories

A symmetric monoidal $\underline{S}$-indexing of a symmetric monoidal category $\mathcal{V}$ is a pseudofunctor

$$\mathbb{V} : \underline{S}^{op} \longrightarrow \underline{\mathrm{SymMonCat}}$$

strong symmetric monoidal functors

where $\underline{S}$ is cartesian monoidal and $\mathbb{V}^1 \cong \mathcal{V}$.

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Non-example (self indexing)

$$\underline{S} = \mathcal{V} \qquad \mathbb{V}^J = \mathcal{V}/J \qquad (X \xrightarrow{x} J) \otimes_J (Y \xrightarrow{y} J) = \,?$$

Comonoids

A (cocommutative) comonoid J is an object J equipped with morphisms

$$d_J : J \longrightarrow J \otimes J \qquad \text{and} \qquad e_J : J \longrightarrow I$$

comultiplication — counit

subject to coassociativity, counitality and cocommutativity laws.

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If $\mathcal{V}$ is cartesian monoidal, then $\underline{\mathrm{Comon}}_{\mathcal{V}} \simeq \mathcal{V}$

Proposition (Fox 1975)

$\underline{\mathrm{Comon}}_{\mathcal{V}}$ is the cofree cartesian monoidal category on $\mathcal{V}$

Comodules

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Proposition

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Comonoid indexing

The comonoid indexing of a symmetric monoidal category $\mathcal{V}$ with nice equalisers is the indexed category

$$\underline{\mathrm{Comod}}_{\mathcal{V}}(-) : \underline{\mathrm{Comon}}_{\mathcal{V}}{}^{op} \longrightarrow \underline{\mathrm{SymMonCat}}$$

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Proposition

If $\mathcal{V}$ is cartesian monoidal, then it is finitely complete and its comonoid indexing is isomorphic with its self indexing.

Monoidal extensivity

A symmetric monoidal category $\mathcal{V}$ is (infinitary) monoidal extensive if it has small coproducts and the functor

$$\sum_{j\in J} : \prod_{j\in J} \underline{\mathrm{Comod}}_{\mathcal{V}}(A_j) \longrightarrow \underline{\mathrm{Comod}}_{\mathcal{V}}\left(\sum_{j\in J} A_j\right)$$

is always an equivalence of categories.

Example (Grunenfelder-Paré 1987) $\underline{\mathrm{Vect}}_{\Bbbk}$

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Example (Grunenfelder-Paré 1987) $\underline{\mathrm{Vect}}_{\mathbb{k}}$

An indexed symmetric monoidal category $\mathbb{V} : \underline{S}^{op} \longrightarrow \underline{\mathrm{SymMonCat}}$ is monoidal extensive if it has indexed coproducts and the functor

$$\Sigma_r : \mathrm{Comod}_{\mathbb{V}J}(A) \longrightarrow \mathrm{Comod}_{\mathbb{V}K}(\Sigma_r A)$$

is always an equivalence of categories.

Universality of comonoid indexing *

$$
\begin{tikzcd}[column sep=huge]
\underline{\mathrm{ExtEqIndSymMonCat}} \arrow[r, bend left, "\text{underlying category}"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{EqSymMonCat}} \arrow[l, bend left, "\text{comonoid indexing}"]
\end{tikzcd}
$$

* work in progress

Conclusion

- Comonoid indexing of nice symmetric monoidal categories generalises self indexing of finitely complete categories

- Taking the self indexing/comonoid indexing is right adjoint to taking the underlying category

- There is a monoidal generalisation of extensivity

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Future work

- Finish checking details for monoidal case
- Check monoidal version of Moens' theorem
- Investigate links to linear dependent types
- Work out link between categories internal to and enriched in a monoidal category via monoidal extensivity