Remarks on enriched protomodularity

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2. Comma objects
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    $(\forall x \in X,\; f(x) \leqslant g(x) \;\Rightarrow\; -f(x) \leqslant -g(x) \;\Rightarrow\; g(x) \leqslant f(x) \;\Rightarrow\; g \preccurlyeq f)$

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$$\begin{tikzcd}
f/g \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\preccurlyeq"] & Z \arrow[d, "g"] \\
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$$\begin{tikzcd}
A \arrow[rrd, bend left, "\beta"] \arrow[rd, dotted, "\lambda=\langle\alpha,\beta\rangle"'] \arrow[rdd, bend right, "\alpha"'] & & \\
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(C1) $f\pi_1 \preccurlyeq g\pi_2$

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(C3) $\begin{cases} \alpha \preccurlyeq \alpha', \beta \preccurlyeq \beta' \\ f\alpha \preccurlyeq g\beta \\ f\alpha' \preccurlyeq g\beta' \end{cases} \;\Rightarrow\; \lambda \preccurlyeq \lambda'$

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$$\begin{tikzcd}
A \arrow[drr, bend left, "\beta"'] \arrow[drr, bend left=40, "\beta'"] \arrow[ddr, bend right=40, "\alpha"'] \arrow[ddr, bend right=25, "\alpha'"] \arrow[dr, dotted, "\lambda"'] \arrow[dr, dotted, shift left, "\lambda'"] & & \\
& f/g \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\preccurlyeq"] & Z \arrow[d, "g"] \\
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A \arrow[dr, dotted, "\lambda"'] \arrow[dr, dotted, bend left=15, "\lambda'"] \arrow[drr, bend left, "\beta"'] \arrow[drr, bend left=40, "\beta'"] \arrow[ddr, bend right, "\alpha"'] \arrow[ddr, bend right=20, "\alpha'"] & & \\
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- precomma object of (f, g): (C1) + (C2)

Precomma objects in $\mathbb{O}$rdAb

- $$\begin{tikzcd}
(X \times Z, P_{f/g}) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\preccurlyeq"] & Z \arrow[d, "g"] \\
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(X \times Z, \{(0,0)\}) \arrow[rrd, bend left, "\pi_Z"] \arrow[rdd, bend right, "\pi_X"'] \arrow[rd, dotted] & & \\
& f/g \arrow[r] \arrow[d] \arrow[rd, phantom, "\preccurlyeq"] & Z \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

    $f\pi_X \preccurlyeq g\pi_Z$

Precomma objects in $\mathbb{O}$rdAb

- $$\begin{tikzcd} (X \times Z, P_{f/g}) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\preccurlyeq"] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

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$(f\pi_X(0,0) = 0 \leqslant 0 = g\pi_Z(0,0))$

Precomma objects in $\mathbb{O}\mathrm{rdAb}$

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    $f\pi_X \preccurlyeq g\pi_Z$

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    $\Rightarrow \; f/g \cong X \times Z$ (as groups)

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  - **(C1)**, **(C2)** hold: precomma obj

Precomma objects in $\mathbb{O}$rdAb

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    $(f\pi_X(0,0) = 0 \leqslant 0 = g\pi_Z(0,0))$

    $\Rightarrow$ $f/g \cong X \times Z$ (as groups)

  - **(C1)**, **(C2)** hold: precomma obj

  - **(C3)** doesn't hold: not a comma obj

Comma objects on points

- $\mathrm{Pt}_{\boldsymbol{Y}}(\mathbb{C})$ cat of points over $\boldsymbol{Y}$: $\boldsymbol{Z} \underset{t}{\overset{g}{\rightleftarrows}} \boldsymbol{Y}$, $\quad gt = 1_{\boldsymbol{Y}}$

Comma objects on points

- $\mathrm{Pt}_{Y}(\mathbb{C})$ cat of points over Y:

$$\begin{tikzcd}
Z \arrow[r, shift left, "g"] \arrow[d, "\gamma"'] & Y \arrow[l, shift left, "t"] \arrow[ld, shift left, "s"] \\
X \arrow[ru, shift left, "f"] &
\end{tikzcd}, \qquad gt = 1_Y$$

$$fs = 1_Y,\ f\gamma = g,\ \gamma t = s$$

Comma objects on points

- $\mathrm{Pt}_{\boldsymbol{Y}}(\mathbb{C})$ cat of points over $\boldsymbol{Y}$:

$$\begin{tikzcd}
\boldsymbol{Z} \arrow[r, shift left, "\boldsymbol{g}"] \arrow[d, "\gamma"'] & \boldsymbol{Y} \arrow[l, shift left, "\boldsymbol{t}"] \arrow[dl, shift left, "\boldsymbol{s}"] \\
\boldsymbol{X} \arrow[ur, shift left, "\boldsymbol{f}"] &
\end{tikzcd}$$

$\boldsymbol{gt} = 1_{\boldsymbol{Y}}$

$\boldsymbol{fs} = 1_{\boldsymbol{Y}}, \ \boldsymbol{f}\gamma = \boldsymbol{g}, \ \gamma\boldsymbol{t} = \boldsymbol{s}$

- $\boldsymbol{V}_{\alpha} : \ \mathrm{Pt}_{\boldsymbol{Y}}(\mathbb{C}) \longrightarrow \mathrm{Pt}_{\boldsymbol{A}}(\mathbb{C})$ vertical comma object functor

Comma objects on points

- $\mathrm{Pt}_Y(\mathbb{C})$ cat of points over Y:

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Z \arrow[r, shift left, "g"] \arrow[d, "\gamma"'] & Y \arrow[l, shift left, "t"] \arrow[dl, shift left, "s"] \\
X \arrow[ur, shift left, "f"] &
\end{tikzcd}$$

$$gt = 1_Y$$

$$fs = 1_Y,\ f\gamma = g,\ \gamma t = s$$

- $V_\alpha : \mathrm{Pt}_Y(\mathbb{C}) \longrightarrow \mathrm{Pt}_A(\mathbb{C})$ vertical comma object functor

$$\begin{tikzcd}
A \arrow[drr, bend left, "s\alpha"] \arrow[dr, dotted, "\langle 1_A, s\alpha \rangle"'] \arrow[ddr, bend right, "1_A"'] & & \\
& \alpha/f \arrow[r] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\preccurlyeq"] & X \arrow[d, shift right, "f"'] \\
& A \arrow[r, "\alpha"'] & Y \arrow[u, shift right, "s"']
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\alpha/f \arrow[d, shift right, "\pi_1"'] \\
A \arrow[u, shift right, "\langle 1_A, s\alpha \rangle"']
\end{tikzcd}$$

$$\alpha \preccurlyeq fs\alpha \ (= \alpha)$$

Comma objects on points

- $\mathrm{Pt}_{\boldsymbol{Y}}(\mathbb{C})$ cat of points over $\boldsymbol{Y}$:

$$\begin{tikzcd} Z \arrow[r, shift left, "g"] \arrow[d, "\gamma"'] & Y, \arrow[l, shift left, "t"] \arrow[dl, shift left, "s"] \\ X \arrow[ur, shift left, "f"] & \end{tikzcd}$$

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- $V_\alpha : \mathrm{Pt}_{\boldsymbol{Y}}(\mathbb{C}) \longrightarrow \mathrm{Pt}_{\boldsymbol{A}}(\mathbb{C})$ vertical comma object functor

$$\begin{tikzcd} Z \arrow[dr, "\gamma"] \arrow[ddr, shift right, "g"'] & \\ & X \arrow[d, shift right, "f"'] \\ & Y \arrow[u, shift right, "s"'] \arrow[uul, shift right, "t"'] \end{tikzcd} \longmapsto \begin{tikzcd} \alpha/g \arrow[dr, "V_\alpha(\gamma)"] \arrow[ddr, shift right] & \\ & \alpha/f \arrow[d, shift right, "\pi_1"'] \\ & A \arrow[u, shift right, "\langle 1_A{,}s\alpha\rangle"'] \arrow[uul, shift right] \end{tikzcd}$$

Comma objects on points

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\end{tikzcd}, \qquad \begin{array}{l} gt = 1_Y \\ fs = 1_Y,\ f\gamma = g,\ \gamma t = s \end{array}$$

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$$\begin{tikzcd}
Z \arrow[dr, "\gamma"] \arrow[ddr, shift right, "g"'] & & & \alpha/g \arrow[dr, "V_\alpha(\gamma)"] \arrow[ddr, shift right] & \\
& X \arrow[d, shift right, "f"'] \arrow[r, phantom, "\longmapsto"] & \; & & \alpha/f \arrow[d, shift right, "\pi_1"'] \\
& Y \arrow[uul, shift right, "t"'] \arrow[u, shift right, "s"'] & & & A \arrow[uul, shift right] \arrow[u, shift right, "\langle 1_A{,}s\alpha\rangle"']
\end{tikzcd}$$

- $H_\alpha(\, X \underset{s}{\overset{f}{\rightleftarrows}} Y \,) = (\, f/\alpha \rightleftarrows A \,)$ horizontal comma object functor

Comma objects on points

- $\mathrm{Pt}_{\boldsymbol{Y}}(\mathbb{C})$ cat of points over $\boldsymbol{Y}$:

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Z \arrow[r, shift left, "g"] \arrow[d, "\gamma"'] & Y \arrow[l, shift left, "t"] \arrow[dl, shift left, "s"] \\
X \arrow[ur, shift left, "f"] &
\end{tikzcd}, \qquad \begin{array}{l} gt = 1_Y \\ fs = 1_Y ,\ f\gamma = g ,\ \gamma t = s \end{array}$$

- $V_\alpha : \mathrm{Pt}_Y(\mathbb{C}) \longrightarrow \mathrm{Pt}_A(\mathbb{C})$ vertical comma object functor

$$\begin{tikzcd}
Z \arrow[dr, "\gamma"] \arrow[ddr, shift right, "g"'] & & & \alpha/g \arrow[dr, "V_\alpha(\gamma)"] \arrow[ddr, shift right] & \\
& X \arrow[d, shift right, "f"'] & \longmapsto & & \alpha/f \arrow[d, shift right, "\pi_1"'] \\
& Y \arrow[uul, shift right, "t"'] \arrow[u, shift right, "s"'] & & & A \arrow[uul, shift right] \arrow[u, shift right, "\langle 1_A , s\alpha \rangle"']
\end{tikzcd}$$

V_α , H_α
like α^*

- $H_\alpha \big(X \underset{s}{\overset{f}{\rightleftarrows}} Y \big) = \big(f/\alpha \rightleftarrows A \big)$ horizontal comma object functor

Comma objects vs pbs I

- precomma objects ↔ pullbacks

  comma objects ↔ 2-pullbacks

Comma objects vs pbs I

- precomma objects ↔ pullbacks

  comma objects ↔ 2-pullbacks

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\preccurlyeq"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

Comma objects vs pbs I

- precomma objects $\leftrightarrow$ pullbacks
  comma objects $\leftrightarrow$ 2-pullbacks

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\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\preccurlyeq"] & \bullet \arrow[d] \\
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\end{tikzcd}
\qquad
\begin{tikzcd}
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\bullet \arrow[r] & \bullet
\end{tikzcd}$$

- Glueing comma objs: $\boxed{\mathbf{1}}$ and $\boxed{\mathbf{2}}$ comma objs $\nRightarrow$ $\boxed{\mathbf{1}}\!\boxed{\mathbf{2}}$ comma obj

Comma objects vs pbs I

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  comma objects ↔ 2-pullbacks

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\preccurlyeq"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

- Glueing comma objs: $\boxed{1}$ and $\boxed{2}$ comma objs $\underset{\text{(C2) fails}}{\not\Rightarrow}$ $\boxed{1}\boxed{2}$ comma obj

Comma objects vs pbs I

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$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\preccurlyeq"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

- Glueing ${}^{\text{pre}}$comma objs: $\boxed{1}$ and $\boxed{2}$ ${}^{\text{pre}}$comma objs $\underset{\text{(C2) fails}}{\not\Rightarrow}$ $\boxed{1}\!\boxed{2}$ ${}^{\text{pre}}$comma obj

Comma objects vs pbs I

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$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\preccurlyeq"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

- Glueing $\overset{\text{pre}}{\text{comma}}$ objs: $\boxed{1}$ and $\boxed{2}$ $\overset{\text{pre}}{\text{comma}}$ objs $\underset{\text{(C2) fails}}{\not\Rightarrow}$ $\boxed{1}\boxed{2}$ $\overset{\text{pre}}{\text{comma}}$ obj

- $\mathbb{O}$rdAb:

$$\begin{tikzcd}
U \times X \times Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\preccurlyeq"] & X \times Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\preccurlyeq"] & Z \arrow[d] \\
U \arrow[r] & X \arrow[r] & Y
\end{tikzcd}$$

Comma objects vs pbs I

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- $\mathbb{O}\mathrm{rdAb}$: $\boldsymbol{U \times Z} \not\cong$

$$\begin{tikzcd} \boldsymbol{U \times X \times Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\preccurlyeq"] & \boldsymbol{X \times Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\preccurlyeq"] & \boldsymbol{Z} \arrow[d] \\ \boldsymbol{U} \arrow[r] & \boldsymbol{X} \arrow[r] & \boldsymbol{Y} \end{tikzcd}$$

Comma objects vs pbs II

- $\mathbb{C}$ lex Ord-enriched category with comma objs

Comma objects vs pbs II

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L1.

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowright\ \boxed{1}"] & f/g \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\preccurlyeq\ \boxed{2}"] & Z \arrow[d, "g"] \\
X' \arrow[r, "x"'] & X \arrow[r, "f"'] & Y
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$\boxed{1}$ commutative, $\boxed{2}$ comma obj

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X' \arrow[r, "x"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$\boxed{1}$ commutative, $\boxed{2}$ comma obj

$\boxed{1}\boxed{2}$ comma obj $\Leftrightarrow$ $\boxed{1}$ 2-pb

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$\boxed{1}$ commutative, $\boxed{2}$ $^{\text{pre}}$comma obj

$\boxed{1}\boxed{2}$ comma obj $\Leftrightarrow$ $\boxed{1}$ 2-pb

$\boxed{1}\boxed{2}$ precomma obj $\Leftrightarrow$ $\boxed{1}$ pb

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\end{tikzcd}$$

$\boxed{1}$ commutative, $\boxed{2}$ $^{\text{pre}}$comma obj

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$\boxed{1}\boxed{2}$ precomma obj $\Leftrightarrow$ $\boxed{1}$ pb

L2. $(=$ L1 in $\mathbb{C}^{co})$

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowright \; \boxed{1}"] & Z' \arrow[d, "z"] \\
f/g \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\preccurlyeq \; \boxed{2}"] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$\boxed{1}$ commutative, $\boxed{2}$ comma obj

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P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowright \ \boxed{1}"] & f/g \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\preccurlyeq \ \boxed{2}"] & Z \arrow[d, "g"] \\
X' \arrow[r, "x"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$\boxed{1}$ commutative, $\boxed{2}$ $^{\text{pre}}$comma obj

$\boxed{1}\boxed{2}$ comma obj $\Leftrightarrow$ $\boxed{1}$ 2-pb

$\boxed{1}\boxed{2}$ precomma obj $\Leftrightarrow$ $\boxed{1}$ pb

L2. ($=$ L1 in $\mathbb{C}^{\mathrm{co}}$)

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowright \ \boxed{1}"] & Z' \arrow[d, "z"] \\
f/g \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\preccurlyeq \ \boxed{2}"] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$\boxed{1}$ commutative, $\boxed{2}$ $^{\text{(pre)}}$comma obj

$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ comma obj $\Leftrightarrow$ $\boxed{1}$ 2-pb

$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ precomma obj $\Leftrightarrow$ $\boxed{1}$ pb

Protomodularity

- $\mathcal{X}$ lex is protomodular [Bourn '91]

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    $\boxed{1}$ and $\boxed{1}\boxed{2}$ pbs $\Rightarrow$ $\boxed{2}$ pb
  - $$\begin{tikzcd}
\bullet \arrow[r, "\pi_2"] \arrow[d, shift right] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, shift right, "f"'] \\
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\end{tikzcd}$$

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\bullet \arrow[r] \arrow[d, shift right] & \bullet \arrow[r] \arrow[d, shift right] & \bullet \arrow[d, shift right] \\
\bullet \arrow[r] \arrow[u, shift right] \arrow[ur, phantom, "\boxed{1}"] & \bullet \arrow[r] \arrow[u, shift right] \arrow[ur, phantom, "\boxed{2}"] & \bullet \arrow[u, shift right]
\end{tikzcd}$$

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- $\mathcal{X}$ lex is protomodular [Bourn '91] (less general)
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- Ex: Grp, Ab, (lex) additive cats, $\mathrm{Set}^{\mathrm{op}}$ (dual of any elementary topos)

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  -
  -
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L1. $\boxed{2}$ comma obj. $\boxed{1}\boxed{2}$ comma obj $\Leftrightarrow$ $\boxed{1}$ 2-pb

L2. $\boxed{2}$ comma obj. $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$ comma obj $\Leftrightarrow$ $\boxed{1}$ 2-pb

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  - (i) $V_\alpha \colon \mathrm{Pt}_Y(\mathbb{C}) \to \mathrm{Pt}_A(\mathbb{C})$ conservative, $\forall\ \alpha \colon A \to Y$
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\alpha/f \arrow[r, "\pi_2"] \arrow[d, shift right] \arrow[dr, phantom, "\preccurlyeq"] & X \arrow[d, shift right, "f"'] \\
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(π_2, s) jee

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$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\rotatebox{-90}{$\leqslant$}"] & Y \end{tikzcd} , \quad \forall x \in P_X, f(x) \leqslant g(x)$$

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$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\text{从}"] & Y \end{tikzcd}, \quad \cdots$$ *under construction*

by Profs Peter Johnstone and Maria Manuel Clementino

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  - $V_\alpha : \mathbb{O}\mathrm{rdAb}/Y \rightarrow \mathbb{O}\mathrm{rdAb}/A$ (precomma obj functors) conservative

$$\begin{tikzcd}
A \times X \arrow[r, shift left, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[l, shift left, "\langle 0,1_X \rangle"] \arrow[d, "f"] \\
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$x \in P_X \Rightarrow (0, x) \in P_{\alpha/f}$ because

$(0, x) \in P_A \times P_X$

$\alpha(0) = 0 \leqslant f(x)$

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 & A \times X \arrow[rr, shift left, "\pi_2"] \arrow[d, "\pi_1"] \arrow[ul, green, shift left] & & X \arrow[ll, green, shift left] \arrow[d, "f"] \arrow[ul, green, shift right, "\gamma^{-1}"'] \\
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$$\begin{tikzcd}
(\mathbb{Z}\times\mathbb{Z},0) \arrow[dr, equal] \arrow[drr] \arrow[dd, "1_{\mathbb{Z}\times\mathbb{Z}}"'] & & \\
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(\mathbb{Z}\times\mathbb{Z},0\times\mathbb{N}) \arrow[dr, "1_{\mathbb{Z}\times\mathbb{Z}}"'] \arrow[drr, shift left, "+"] & & \\
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