

Remarks on enriched protomodularity

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20 June 2022

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- 2 Comma objects
- 3 Ord-protomodularity
- 4 Examples

Motivation

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($\forall x \in X, f(x) \leq g(x) \Rightarrow -f(x) \leq -g(x) \Rightarrow g(x) \leq f(x) \Rightarrow g \preceq f$)

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$$X \begin{array}{c} \xrightarrow{0} \\ \lrcorner \\ \xrightarrow{\forall g} \end{array} Y$$

$$\mathbb{Z} \begin{array}{c} \xrightarrow{f(x)=2x} \\ \lrcorner \\ \xrightarrow{g(x)=7x} \end{array} \mathbb{Z}$$

$$(\mathbb{Z}, \mathbb{N}) \begin{array}{c} \xrightarrow{f(x)=9x} \\ \lrcorner \\ \xrightarrow{g(x)=3x} \end{array} (\mathbb{Z}, 3\mathbb{Z})$$

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 f/g & \xrightarrow{\pi_2} & Z \\
 \pi_1 \downarrow & \cong & \downarrow g \\
 X & \xrightarrow{f} & Y
 \end{array}$$

Definition

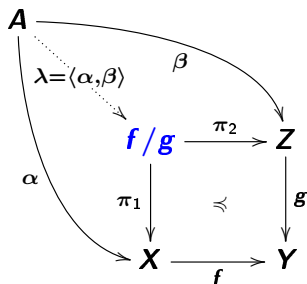
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$$(C1) \quad f\pi_1 \preceq g\pi_2$$

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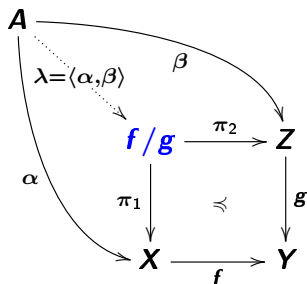


$$(C1) \quad f\pi_1 \cong g\pi_2$$

$$(C2) \quad f\alpha \cong g\beta \Rightarrow \exists! \lambda : \begin{cases} \pi_1 \lambda = \alpha \\ \pi_2 \lambda = \beta \end{cases}$$

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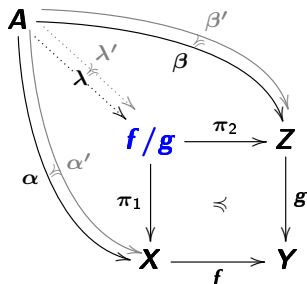
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$$(C3) \quad \begin{cases} \alpha \cong \alpha', \beta \cong \beta' \\ f\alpha \cong g\beta \\ f\alpha' \cong g\beta' \end{cases} \Rightarrow \lambda \cong \lambda'$$

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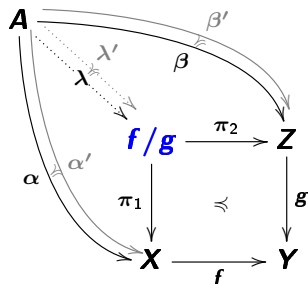
$$(C1) \quad f\pi_1 \preccurlyeq g\pi_2$$

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- precomma object of (f, g) : (C1) + (C2)

Precomma objects in $\mathbb{O}rdAb$

$$\begin{array}{ccc}
 (X \times Z, P_{f/g}) & \xrightarrow{\pi_2} & Z \\
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 X & \xrightarrow{f} & Y
 \end{array}$$

$$P_{f/g} = \{(x, z) \in P_X \times P_Z : f(x) \leq g(z)\}$$

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• $(X \times Z, \{(0, 0)\})$

$$\begin{array}{ccccc}
 (X \times Z, \{(0, 0)\}) & & \xrightarrow{\pi_Z} & & Z \\
 & \searrow \text{dotted} & & \xrightarrow{f/g} & \downarrow g \\
 & & & X & \xrightarrow{f} & Y \\
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$f\pi_X \cong g\pi_Z$
 $(f\pi_X(0, 0) = 0 \leq 0 = g\pi_Z(0, 0))$

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 f/g & \xrightarrow{\quad} & Z \\
 \downarrow & \cong & \downarrow g \\
 X & \xrightarrow{f} & Y \\
 \uparrow \pi_X & & \\
 (X \times Z, \{(0, 0)\}) & &
 \end{array}$$

$$f\pi_X \cong g\pi_Z$$

$$(f\pi_X(0, 0) = 0 \leq 0 = g\pi_Z(0, 0))$$

$$\Rightarrow f/g \cong X \times Z \text{ (as groups)}$$

Precomma objects in $\mathbb{O}rdAb$

• $(X \times Z, P_{f/g}) \xrightarrow{\pi_2} Z$ $P_{f/g} = \{(x, z) \in P_X \times P_Z : f(x) \leq g(z)\}$

$$\begin{array}{ccc}
 & & \downarrow g \\
 & \cong & \\
 & & Y \\
 \pi_1 \downarrow & & \uparrow f \\
 X & \xrightarrow{\quad} &
 \end{array}$$

- $(X \times Z, \{(0, 0)\})$

$$\begin{array}{ccc}
 & & \downarrow g \\
 & \cong & \\
 & & Y \\
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 X & \xrightarrow{\quad} &
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$f/g \xrightarrow{\quad} Z$
 \downarrow
 X

$f \pi_X \cong g \pi_Z$
 $(f \pi_X(0, 0) = 0 \leq 0 = g \pi_Z(0, 0))$
 $\Rightarrow f/g \cong X \times Z$ (as groups)

- (C1), (C2) hold: precomma obj

Precomma objects in $\mathbb{O}rdAb$

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$$\begin{array}{ccc}
 & \xrightarrow{\pi_2} & Z \\
 \pi_1 \downarrow & \cong & \downarrow g \\
 X & \xrightarrow{f} & Y
 \end{array}$$

- $(X \times Z, \{(0, 0)\})$

$$\begin{array}{ccc}
 & \xrightarrow{\pi_Z} & Z \\
 \text{dotted arrow} & \searrow & \downarrow g \\
 & f/g & \rightarrow & Z \\
 \downarrow & \cong & \downarrow g \\
 \pi_X \searrow & & X & \xrightarrow{f} & Y
 \end{array}$$

$f\pi_X \cong g\pi_Z$
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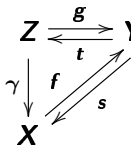
- (C1), (C2) hold: precomma obj
- (C3) doesn't hold: not a comma obj

Comma objects on points

- $\text{Pt}_Y(\mathbb{C})$ cat of **points over Y** : $Z \begin{array}{c} \xrightarrow{g} \\ \xleftarrow{t} \end{array} Y, \quad gt = 1_Y$

Comma objects on points

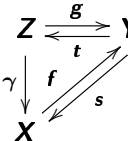
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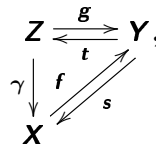
$$fs = 1_Y, f\gamma = g, \gamma t = s$$

Comma objects on points

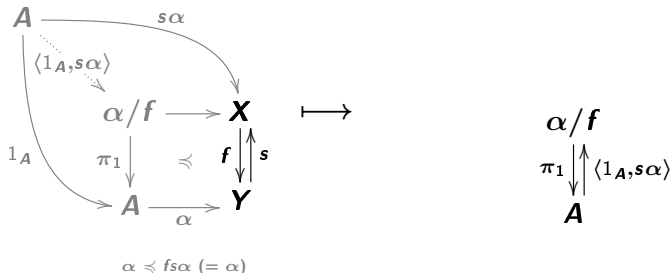
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- $V_\alpha : \text{Pt}_Y(\mathbb{C}) \longrightarrow \text{Pt}_A(\mathbb{C})$ **vertical comma object functor**

Comma objects on points

- $\text{Pt}_Y(\mathbb{C})$ cat of **points over Y** : $Z \begin{matrix} \xrightarrow{g} \\ \xleftarrow{t} \end{matrix} Y, \quad \begin{matrix} gt = 1_Y \\ fs = 1_Y, f\gamma = g, \gamma t = s \end{matrix}$


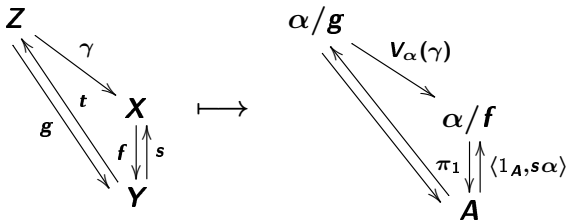
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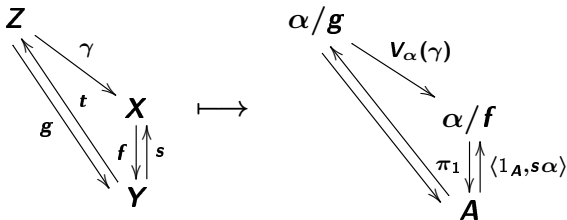
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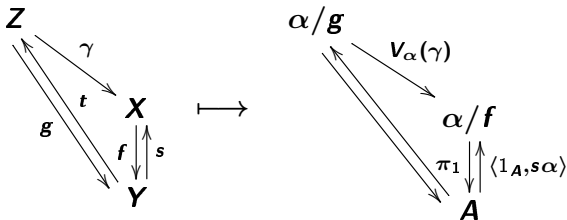


- $H_\alpha(X \begin{matrix} \xrightarrow{f} \\ \xleftarrow{s} \end{matrix} Y) = (f/\alpha \rightleftarrows A)$ **horizontal comma object functor**

Comma objects on points

- $\text{Pt}_Y(\mathbb{C})$ cat of **points over Y** : $Z \begin{matrix} \xrightarrow{g} \\ \xleftarrow{t} \end{matrix} Y, \quad gt = 1_Y$
 $\begin{matrix} \gamma \downarrow \\ X \end{matrix} \begin{matrix} \nearrow f \\ \searrow s \end{matrix}$ $fs = 1_Y, f\gamma = g, \gamma t = s$

- $V_\alpha : \text{Pt}_Y(\mathbb{C}) \longrightarrow \text{Pt}_A(\mathbb{C})$ **vertical comma object functor**



V_α, H_α
like α^*

- $H_\alpha(X \begin{matrix} \xrightarrow{f} \\ \xleftarrow{s} \end{matrix} Y) = (f/\alpha \begin{matrix} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{matrix} A)$ **horizontal comma object functor**

Comma objects vs pbs I

- precomma objects \leftrightarrow pullbacks
 comma objects \leftrightarrow 2-pullbacks

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Comma objects vs pbs I

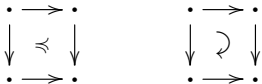
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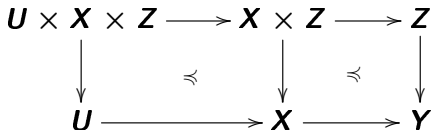
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- Glueing ^{pre}comma objs: $\boxed{1}$ and $\boxed{2}$ ^{pre}comma objs $\not\Rightarrow$ $\boxed{1 \mid 2}$ ^{pre}comma obj
 (C2) fails

• OrdAb: $U \times Z \not\cong U \times X \times Z \longrightarrow X \times Z \longrightarrow Z$

$$\begin{array}{ccccc}
 U \times Z & \longrightarrow & U \times X \times Z & \longrightarrow & X \times Z & \longrightarrow & Z \\
 \downarrow & & \downarrow & \Downarrow & \downarrow & \Downarrow & \downarrow \\
 U & \longrightarrow & X & \longrightarrow & Y & &
 \end{array}$$

Comma objects vs pbs II

- \mathbb{C} lex Ord-enriched category with comma objs

Comma objects vs pbs II

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L1.

$$\begin{array}{ccccc}
 P & \xrightarrow{p_2} & f/g & \xrightarrow{\pi_2} & Z \\
 \downarrow p_1 & & \downarrow \pi_1 & & \downarrow g \\
 X' & \xrightarrow{x} & X & \xrightarrow{f} & Y
 \end{array}$$

A curved arrow labeled \cong points from the top row to the bottom row, indicating a natural isomorphism between the two rows.

Boxed numbers 1 and 2 are placed below the arrows x and f respectively.

Box 1 is commutative, Box 2 is comma obj

Comma objects vs pbs II

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 & \boxed{1} & & \boxed{2} & \\
 & \text{comma obj} & & \text{2-pb} &
 \end{array}$$

$\pi_1 \circ x = p_1$ (indicated by a curved arrow) and $f \circ x = p_1$ (indicated by \cong).

$\boxed{1}$ commutative, $\boxed{2}$ comma obj

$\boxed{1} \boxed{2}$ comma obj \Leftrightarrow $\boxed{1}$ 2-pb

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A curved arrow labeled \cong points from the top row to the bottom row.
 A box labeled **1** is placed below the arrow x .
 A box labeled **2** is placed below the arrow f .
 A symbol \cong is placed between the two boxes.

$\boxed{1}$ commutative, $\boxed{2}^{\text{pre}}$ comma obj

$\boxed{1} \boxed{2}$ comma obj \Leftrightarrow $\boxed{1}$ 2-pb

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 \end{array}$$

Diagram description: A commutative square with a triangle. The top row is $P \xrightarrow{p_2} f/g \xrightarrow{\pi_2} Z$. The bottom row is $X' \xrightarrow{x} X \xrightarrow{f} Y$. A vertical arrow p_1 goes from P to X' . A vertical arrow π_1 goes from f/g to X . A vertical arrow g goes from Z to Y . A curved arrow \hookrightarrow goes from x to π_1 . A curved arrow \cong goes from f to g . A box labeled '1' is next to x , and a box labeled '2' is next to f .

$\boxed{1}$ commutative, $\boxed{2}^{\text{pre}}$ comma obj

$\boxed{1} \boxed{2}$ comma obj \Leftrightarrow $\boxed{1}$ 2-pb

$\boxed{1} \boxed{2}$ precomma obj \Leftrightarrow $\boxed{1}$ pb

L2. (= L1 in \mathbb{C}^{co})

$$\begin{array}{ccc}
 P & \xrightarrow{p_2} & Z' \\
 \downarrow p_1 & \hookrightarrow & \downarrow z \\
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$\boxed{1}$ commutative, $\boxed{2}$ comma obj

Comma objects vs pbs II

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Diagram annotations: A curved arrow from P to X' is labeled with a box containing '1'. A curved arrow from f/g to X is labeled with a box containing '2'. A curved arrow from Z to Y is labeled with a box containing '2'.

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 P & \xrightarrow{p_2} & Z' \\
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 X & \xrightarrow{f} & Y
 \end{array}$$

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$\boxed{1}$ commutative, $\boxed{2}^{\text{(pre)}}$ comma obj

$\boxed{\frac{1}{2}}$ comma obj \Leftrightarrow $\boxed{1}$ 2-pb

$\boxed{\frac{1}{2}}$ precomma obj \Leftrightarrow $\boxed{1}$ pb

- 1 Introduction
- 2 Comma objects
- 3 Ord-protomodularity**
- 4 Examples

Protomodularity

- \mathcal{X} lex is protomodular [Bourn '91]

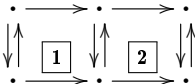
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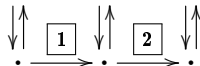
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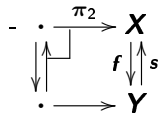
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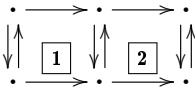
1 and 1 2 pbs \Rightarrow 2 pb



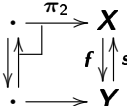
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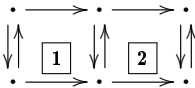
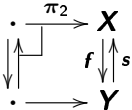
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- 

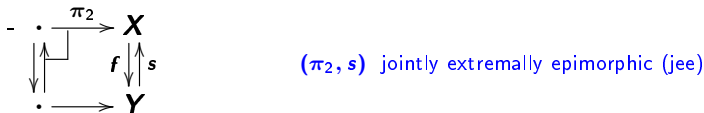
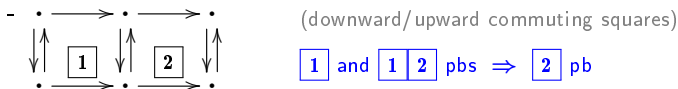
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- Ex: Grp, Ab, (lex) additive cats, Set^{op} (dual of any elementary topos)

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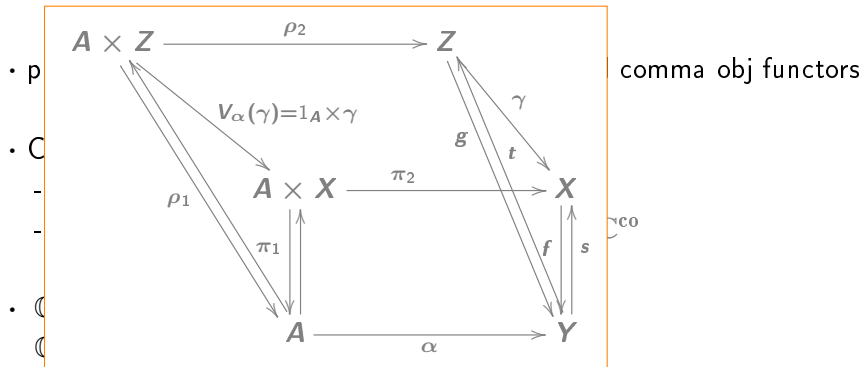
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 & & \downarrow s & & \downarrow t \\
 & & \mathbf{Y} & & \mathbf{V}
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L1. $\boxed{2}$ comma obj. $\boxed{1|2}$ comma obj $\Leftrightarrow \boxed{1}$ 2-pb

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$$\begin{array}{ccc}
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- 1 Introduction
- 2 Comma objects
- 3 Ord-protomodularity
- 4 Examples**

Examples I

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$$\mathbf{X} \begin{array}{c} \xrightarrow{f} \\ \Downarrow \lambda \\ \xrightarrow{g} \end{array} \mathbf{Y}, \quad \dots \text{ under construction}$$

by Profs Peter Johnstone and Maria Manuel Clementino

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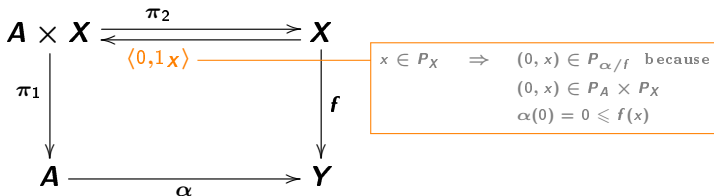
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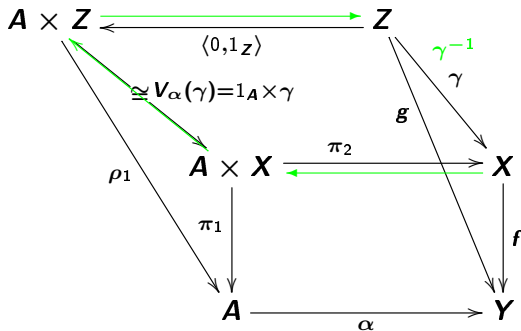
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 \downarrow \rho_1 & \searrow \cong V_\alpha(\gamma) = 1_{\mathbf{A}} \times \gamma & \downarrow g \\
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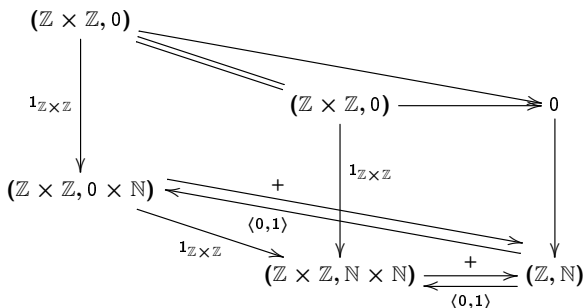
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 - $\Rightarrow V_\alpha: Pt_{\mathbf{Y}}(\mathbb{O}rdAb) \rightarrow Pt_{\mathbf{A}}(\mathbb{O}rdAb) \dots$ conservative

Examples II

- $\mathbb{O}rdAb$: “Ord-protomodular”, **not** “Ord-coprotomodular” (wrt precomma objs)
 - $V_\alpha: \mathbb{O}rdAb/\mathbf{Y} \rightarrow \mathbb{O}rdAb/\mathbf{A}$ (precomma obj functors) conservative
 - $\Rightarrow V_\alpha: Pt_{\mathbf{Y}}(\mathbb{O}rdAb) \rightarrow Pt_{\mathbf{A}}(\mathbb{O}rdAb) \dots$ conservative
 - $H_\alpha: Pt_{\mathbf{Y}}(\mathbb{O}rdAb) \rightarrow Pt_{\mathbf{A}}(\mathbb{O}rdAb)$ (precomma obj functors) **not** conservative

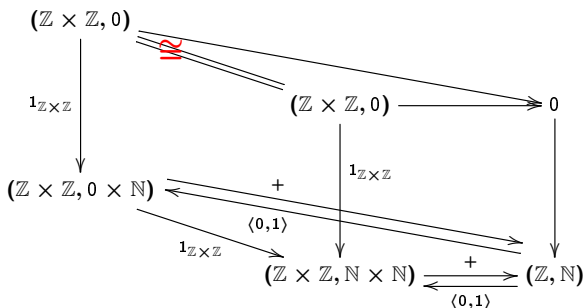
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