The Topological μ-Calculus

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The perfect core is an example of a **topological fixed point.**

Unimodal language

Modal language $\mathcal{L}_{\Diamond}$:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi$$

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Usual abbreviations:

- $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$
- $\Diamond\varphi := \neg\Box\neg\varphi$

Topological c-semantics of modal logic

If $X = (X, c)$ is a topological space, a topological model is a tuple $(X, [\![\cdot]\!])$ where $[\![\cdot]\!] : \mathcal{L}_{\mu}^{\Diamond} \to 2^{X}$ is such that

$$[\![\Box\varphi]\!] = i\,[\![\varphi]\!] \qquad [\![\Diamond\varphi]\!] := c\,[\![\varphi]\!]$$

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Modal logic S4:

- $\Box(p \to q) \to (\Box p \to \Box q)$
- $\Box p \to p$
- $\Box p \to \Box\Box p$
- $\dfrac{\varphi}{\Box\varphi}$

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If $(W, \sqsubseteq)$ is a Kripke frame where $\sqsubseteq$ is a preorder, the upwards-closed sets form a topology $\mathcal{U}_{\sqsubseteq}$ on W.

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3. *Any crowded metric space (Rasiowa and Sikorski, 1960's)*

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Example: Transitive closure:

$$\Diamond^*\varphi := \mu p.(\varphi \vee \Diamond p)$$

Results for the μ-calculus

Axiomatization of μ-K4: Extend the logic K by

- $\varphi(\mu p.\varphi(p)) \to \mu p.\varphi(p)$
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The μ-calculus has the finite model property over the class of Kripke frames.

Theorem (Walukiewicz 1995)

The μ-calculus is sound and complete for the class of Kripke frames.

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Proof idea.

Let φ^* be the result of replacing $\Diamond$ by $\Diamond^*$, and apply Walukiewicz's theorem. $\square$

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Theorem (Goldblatt and Hodkinson, 2016)

μ-S4 *is sound and strongly complete for the class of topological spaces.*

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Theorem (D'Agostino and Lenzi, 2010)

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Corollary

Every formula of the μ-calculus is topologically equivalent to an alternation-free one.

The tangled closure

Define

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3. Topologically, $[\![\Diamond^\infty\{A_1,\ldots,A_n\}]\!]$ is the largest subspace in which every A_i is dense.

Universality of tangle

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Every formula of the μ-calculus is equivalent to a formula in $\mathcal{L}^{\Diamond}_{\Diamond\infty}$ over the class of topological spaces.

Tangled modal logic

Define $S4^\infty$ by adding, for $\Phi = \{\varphi_1, \ldots, \varphi_n\}$,

$$\Diamond^\infty \Phi \to \bigwedge_i \Diamond\Big(\varphi_i \wedge \Diamond^\infty \Phi\Big) \qquad \frac{\theta \to \bigwedge_i \Diamond\Big(\varphi_i \wedge \theta\Big)}{\theta \to \Diamond^\infty \Phi}$$

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Theorem (F-D, 2011)

$S4^\infty$ *is sound and complete for*

- *the class of all topological spaces*
- *the class of all finite topological spaces*

Topological d-semantics

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Warning: From now on, $\Diamond$ is d, $\diamonddot$ is c.

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Theorem (F-D and Iliev, 2018)

$\mathcal{L}^{\diamonddot}$ *and* $\mathcal{L}^{\Diamond}_{\mu}$ *are both exponentially more succinct than* $\mathcal{L}^{\Diamond}$

T_D spaces

Fact: K4 is not sound for the class of topological spaces!

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- $T_0 \supseteq T_D \supseteq T_1 \supseteq T_2$

So, $\mathbb{R}$, $\mathbb{Q}$, the Cantor space, ... are all T_D.

K4 frames as T_D spaces

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If $(W, \sqsubset)$ is any Kripke frame, its tree unwinding can thus be seen as a T_D space.

Completeness of K4

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Theorem
GL *is the logic of finite T_D spaces:*

$$\Box(\Box p \to p) \to \Box p$$

Basic results

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Theorem

Any μ-calculus formula is equivalent to an alternation-free formula over the class of T_D spaces.

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Theorem

Every formula of the μ-calculus is equivalent to a formula in $\mathcal{L}^{\Diamond}_{\Diamond^{\infty}}$ over the class of T_D spaces with Cantor derivative.

Tangled derivative in K4 frames

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Topologically, $[\![\Diamond^\infty\{A_1, \ldots, A_n\}]\!]$ is the largest subspace S in which every A_i is **strictly** dense:

$$S \subseteq d(S \cap A_i)$$

Tangled closure vs. derivative

Not equivalent in general:

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Goldblatt and Hodkinson, 2016: Over T_D spaces,

$$\odot^{\infty}\Phi \equiv \bigwedge \Phi \vee \Diamond \bigwedge \Phi \vee \Diamond^{\infty}\Phi$$

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Note that *Kur* need not be **sound** for X!

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do not follow from classic μ-calculus results!

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Definition (Fine 1985)

Say that T is φ**-final** if $\varphi \in T$ and whenever $S \sqsupseteq T$ and $\varphi \in S$, it follows that $T \sqsupseteq S$.

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Definition (Fine 1985)

Say that T is φ**-final** if $\varphi \in T$ and whenever $S \sqsupseteq T$ and $\varphi \in S$, it follows that $T \sqsupseteq S$.

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Final submodel: $\mathcal{M}_c^\Sigma = (W_c^\Sigma, \sqsubset_c^\Sigma, [\![\cdot]\!]_c^\Sigma)$ is the submodel of Σ-final theories.

Truth lemma for the final submodel

Lemma (Σ-Final Truth Lemma)

Let Σ be finite and closed under subformulas (and a few other operations, such as single negation).

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Cofinal subframe logics

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This includes μ-S4, μ-K4, and many other examples.

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4. *The logic* μ-wK4T$_0$ *(which I won't define here) is sound and complete for the class of* T_0 *spaces with topological derivative.*

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Question: Is the tangled fragment expressively complete?

Expressive incompleteness

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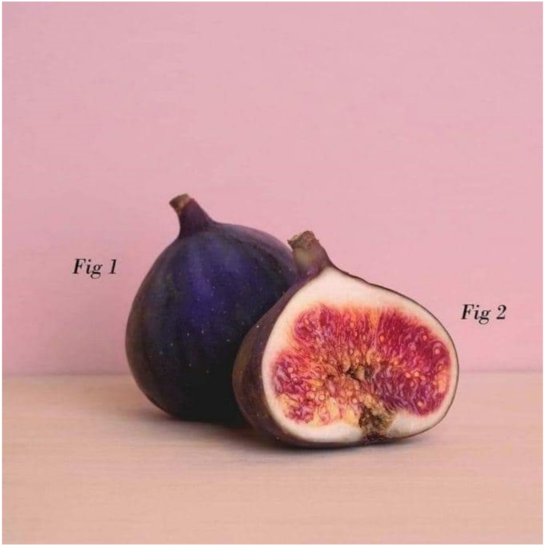
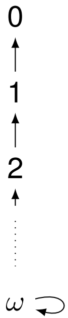


Fig 1
Fig 2

Undefinability of $\Diamond^\infty\{\top\}$ in $\mathcal{L}^{\Diamond}_{\dot{\Diamond}\infty}$



The hybrid tangle

Given $\Phi = (\varphi_1, \ldots, \varphi_n)$,

$$\blacklozenge^{\infty}\Phi := \nu p. \bigvee_{i \leq n} \Big(\odot \, (\varphi_i \wedge \blacklozenge^{\infty}\Phi) \wedge \bigwedge_{j \neq i} \Diamond(\varphi_j \wedge \blacklozenge^{\infty}\Phi) \Big)$$

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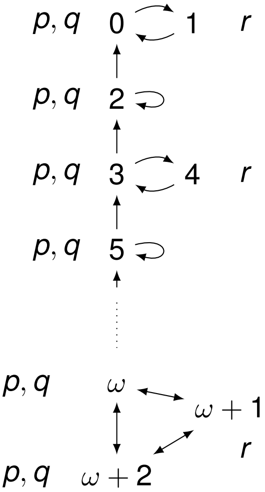

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Obrigado!