Dependence logic and team semantics

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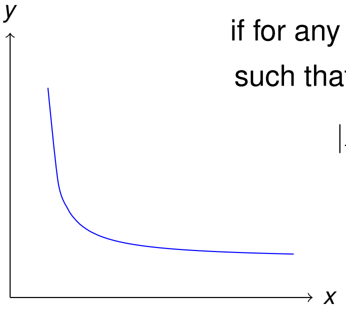
Motivating example (due to Hodges 1997)

Let I be a subset of $\mathbb{R}$.

Definition:

A function $f : I \to \mathbb{R}$ is said to be continuous on I if for any $x_0 \in I$, for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $x \in I$,

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \epsilon.$$



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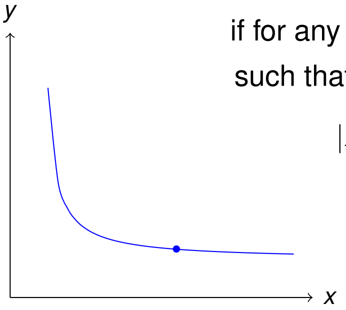

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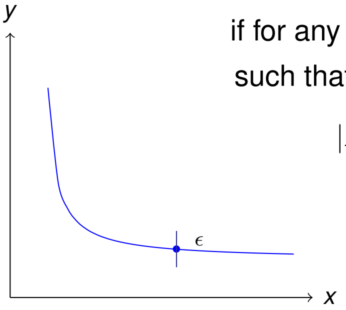

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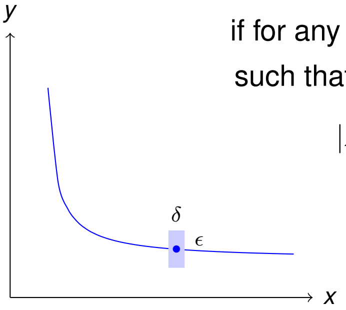

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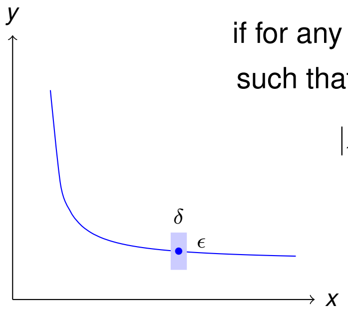

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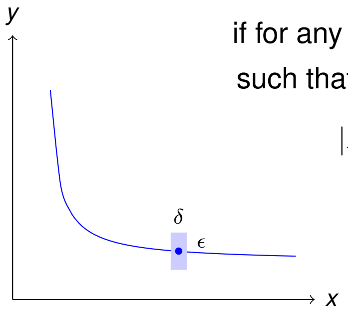

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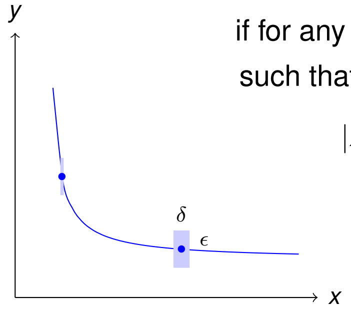

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A function $f : I \to \mathbb{R}$ is said to be ~~continuous~~ uniformly continuous on I if for any $x_0 \in I$, for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $x \in I$,

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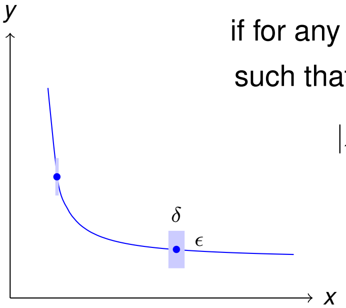


continuity: $\forall x_0 \forall \epsilon \exists \delta \forall x \phi$

Let I be a subset of $\mathbb{R}$.

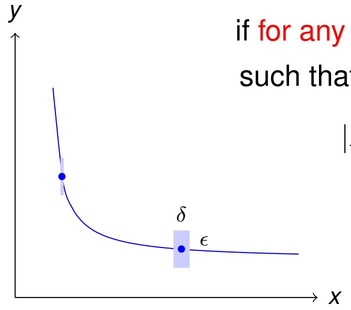
Definition:

A function $f : I \to \mathbb{R}$ is said to be uniformly continuous on I

if for any $x_0 \in I$, for any $\epsilon > 0$ there exists $\delta > 0$ which does not depend on x_0

such that for any $x \in I$,

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \epsilon.$$



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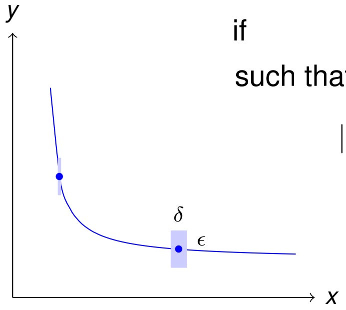

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uniform continuity: $\forall \epsilon \exists \delta \forall x_0 \forall x \phi$

Characterizing dependencies among quantified variables

$$\forall u \, \exists v \, \forall x \, \exists y \, \phi$$

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Henkin Quantifiers (1961):

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meaning $\exists f \exists g \forall u \forall x \phi(u, x, f(u)/v, g(x)/y)$

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Independence-Friendly Logic (Hintikka and Sandu, 1989):

$$\forall u \exists v \forall x \exists y/\{u\} \phi$$

imperfect information game

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Dependence logic (Väänänen 2007): $\forall u \exists v \forall x \exists y (\phi \wedge =(x, y))$

Dependence atoms (Väänänen 2007)

"*x* completely determines *y*"

$=(x, y)$

- iff for all $s, s' \in X$, $s(\vec{x}) = s'(\vec{x})$.

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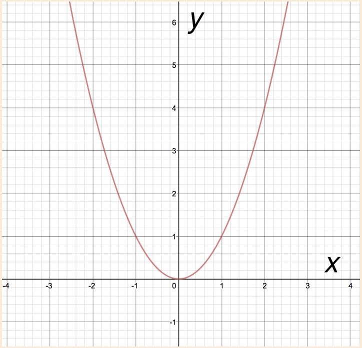

$y = f(x) = x^2$

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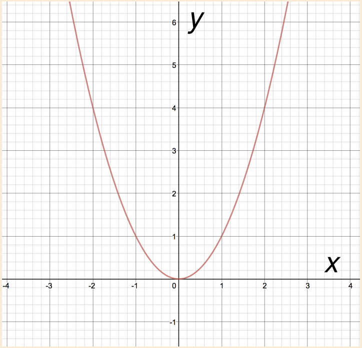
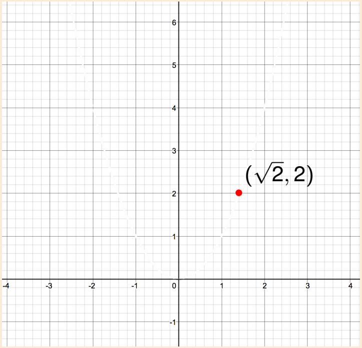


$y = f(x) = x^2$

	x	y	z
s	$\sqrt{2}$	2	0

Given a model M, and an assignment $s : \mathrm{Var} \to M$,

$M \models_s$ "x completely determines y"??



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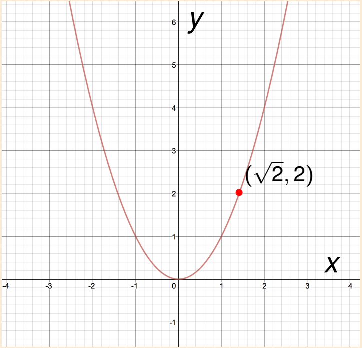

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s_1	$\sqrt{2}$	2	1
s_2	-2	4	$\sqrt{2}$
s_3	-2	4	2
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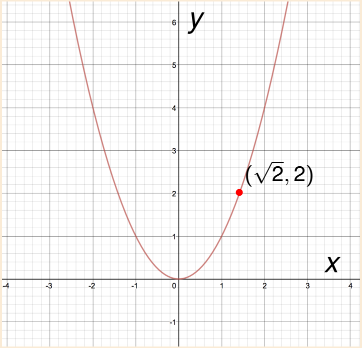

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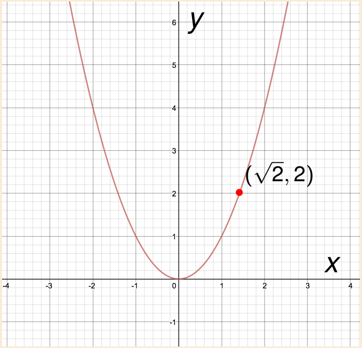

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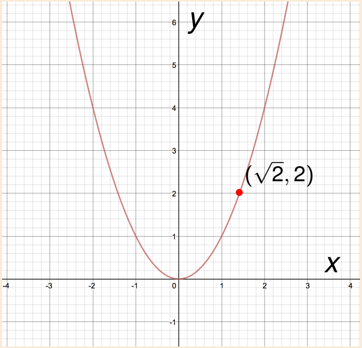

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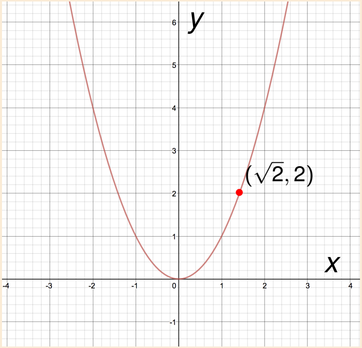

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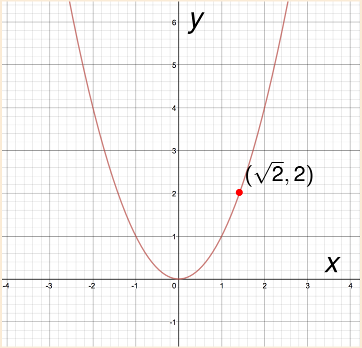

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	w	x	y
s_0	c	a	b
s_1	a	a	d
s_2	a	a	c
s_3	b	a	e
s_4	c	a	a

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- $M \models_X =(\langle\rangle, \vec{x})$ iff for all $s, s' \in X$, $s(\vec{x}) = s'(\vec{x})$.

Dependence atoms (Väänänen 2007) & team semantics (Hodges 1997)

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- Constancy atom: $M \models_X =\!(\vec{x})$ iff for all $s, s' \in X$, $s(\vec{x}) = s'(\vec{x})$.

Connection with database theory

	x	y	z	v
s_0	c	d	a	a
s_1	c	d	a	b
s_2	a	e	c	c
s_3	a	e	c	d

- A team can be viewed as a relational database.
- Dependence atoms $=\!(\vec{x}, \vec{y})$ correspond exactly to functional dependencies $\vec{x} \rightarrow \vec{y}$ in database theory
- Armstrong's Axioms (1974) for functional dependencies:
  - $=\!(\vec{x}, \vec{x})$ (identity)
  - $=\!(\vec{x}\vec{y}, \vec{z})$ implies $=\!(\vec{y}\vec{x}, \vec{z})$ (commutativity)
  - $=\!(\vec{x}\vec{x}, \vec{y})$ implies $=\!(\vec{x}, \vec{y})$ (contraction)
  - $=\!(\vec{y}, \vec{z})$ implies $=\!(\vec{x}\vec{y}, \vec{z})$ (weakening)
  - $=\!(\vec{x}, \vec{y})$ and $=\!(\vec{y}, \vec{z})$ imply $=\!(\vec{x}, \vec{z})$ (transitivity)

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- First-order logic (FO):

$$\alpha ::= t = t' \mid R\vec{t} \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \exists x\alpha \mid \forall x\alpha$$

- Dependence logic (Väänänen 2007):

$$\text{first-order logic} + {=}(\vec{x}, \vec{y})$$

- First-order logic (FO):

  $\alpha ::= t = t' \mid R\vec{t} \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \exists x \alpha \mid \forall x \alpha$

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- Dependence logic (Väänänen 2007):

  $\phi ::= \alpha \mid \neg\alpha \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x\phi \mid \forall x\phi \mid {=}(\vec{x}, \vec{y})$

  where α is an FO-formula $\exists f$

Team semantics

Let X be a team, i.e., a set of assignments $s : \mathrm{Var} \to M$.

- $M \models_X =(\vec{x}, \vec{y})$ iff for all $s, s' \in X$: $s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$, whenever α is a first-order formula
- $M \models_X \neg\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$, whenever α is a first-order formula
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.
- $M \models_X \exists v$ iff
- $M \models_X \forall v \phi$ iff $M \models_{X(M/v)} \phi$, where $X(M/v) = \{s(a/v) \mid s \in X \,\&\, a \in M\}$.

x	y	z
3	4	5
2	3	0
1	2	3
0	1	0

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x	y	z
3	4	5
2	3	0
1	2	3
0	1	0

$x < y$

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$M \not\models_X x < y$

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x	y	z
3	4	5
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1	2	3
0	1	0

$M \not\models_X x < y$

$M \not\models_X \neg x < y$

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x	y	z
3	4	5
2	0	0
1	2	3
0	1	0

$\alpha \vee \beta$

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x	y	z	
3	4	5	α
2	0	0	α
1	2	3	α, β
0	1	0	β

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x	y	z
3	4	5
2	0	0
1	2	3
0	1	0

v
0
1
2
3
⋮

M

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Empty team property: $M \models_\emptyset \phi$.

Downward closure: $M \models_X \phi$ and $Y \subseteq X \Longrightarrow M \models_Y \phi$.

For every formula α of the standard first-order logic,

Union closure: $M \models_{X_i} \alpha$ for all $i \in I \neq \emptyset$, then $M \models_{\bigcup_{i \in i} X_i} \alpha$.

Flatness: $M \models_X \alpha \iff \forall s \in X : M \models_{\{s\}} \alpha \iff \forall s \in X : M \models_s \alpha$.

$x \perp y$

Independence atoms and independence logic FO(⊥) (Grädel, Väänänen 2013)

$x \perp y$

$x \perp y$

- $M \models_X \vec{x} \perp \vec{y}$ iff for all $s, s' \in X$, there exists $s'' \in X$ such that $s''(\vec{x}) = s(\vec{x})$ and $s''(\vec{y}) = s'(\vec{y})$.
- $M \models_X \vec{x} \perp_{\vec{z}} \vec{y}$ iff for all $s, s' \in X$ s.t. $s(\vec{z}) = s'(\vec{z})$, there exists $s'' \in X$ s.t. $s''(\vec{z}) = s(\vec{z})$, $s''(\vec{x}) = s(\vec{x})$ and $s''(\vec{y}) = s'(\vec{y})$.

$x \perp y$

x	y	z
a	b	e
c	d	d
c	b	e
a	d	a

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$x \perp y$

x	y	z
a	b	e
c	d	d
c	b	e
a	d	a

$x \perp_z y$

x	y	z
a	b	e
c	d	e
c	b	e
a	d	e

... correspond to embedded multivalued dependencies $\vec{z} \twoheadrightarrow \vec{x} \mid \vec{y}$ in database theory

Fact: $=(x, y) \equiv y \perp_x y$, thus $\mathrm{FO}(=(\ldots)) \lneq \mathrm{FO}(\perp)$ (i.e., $\mathrm{FO} + \vec{x} \perp_{\vec{z}} \vec{y}$).

Expressing dependencies and swapping quantifiers

The following dependence relation:

$$\forall u \, \exists v \, \forall x \, \exists y \, \phi$$

can be expressed in dependence logic as

- $\forall u \exists v \forall x \exists y (\phi \wedge {=}(x, y))$
- or $\forall u \exists v \forall x \exists y (\phi \wedge {=}(x, y) \wedge {=}(u, v))$ (to be more rigorous),
- or even $\forall u \forall x \exists v \exists y (\phi \wedge {=}(x, y) \wedge {=}(u, v))$.

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Expressing dependencies and swapping quantifiers

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$$\forall u \, \exists v \, \forall x \, \overset{\frown}{\exists y} \, \phi$$

can be expressed in dependence logic as

- $\forall u \exists v \forall x \exists y (\phi \wedge =(x, y))$
- or $\forall u \exists v \forall x \exists y (\phi \wedge =(x, y) \wedge =(u, v))$ (to be more rigorous),
- or even $\forall u \forall x \exists v \exists y (\phi \wedge =(x, y) \wedge =(u, v))$.

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Lemma. For any formula ϕ of FO($=$(...)),

$$\exists y \forall x \phi(x, y, \vec{v}) \equiv \forall x \exists y ({=}(\vec{v}, y) \wedge \phi).$$

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Lemma. For any formula ϕ of FO($=(\ldots)$),

$$\exists y \forall x \phi(x, y, \vec{v}) \equiv \forall x \exists y ({=}(\vec{v}, y) \wedge \phi).$$

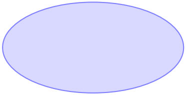
Prop. For any formula ϕ of FO($=(\ldots)$), we have that

$$\phi \equiv \forall \vec{x} \exists \vec{y} \theta$$

for some quantifier-free formula θ.

Pf. First transform ϕ into an equivalent formula in prenex normal form $Q_1 x_1 \ldots Q_n x_n \theta$, where each $Q_i \in \{\forall, \exists\}$ and θ is quantifier-free. Then apply Lemma exhaustively. □

$|M| = \infty$



- An existential second-order (ESO) sentence:
  $\exists f \exists v \forall x_0 \forall x_1 \big((f(x_0) = f(x_1) \to x_0 = x_1) \wedge f(x_0) \neq v \big)$
- An FO(=(...))-sentence:

$$\phi_\infty := \exists v \forall x \exists y ({=}(x, y) \wedge {=}(y, x) \wedge (v \neq y))$$

Defining infinity

$$|M| = \infty \quad \text{iff} \quad \exists f :$$

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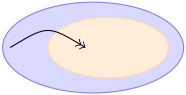
$|M| = \infty$ iff $\exists f$: 

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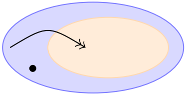
$|M| = \infty$ iff $\exists f$: 

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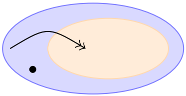
$|M| = \infty$ iff $\exists f$: 

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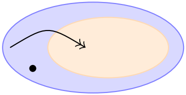
$|M| = \infty$ iff $\exists f$: 

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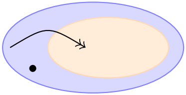
$|M| = \infty$ iff $\exists f$: 

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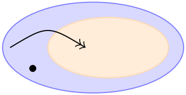
$|M| = \infty$ iff $\exists f$: 

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Theorem (Väänänen 2007)

For any ESO-sentence ϕ, there is an FO(=(...))*-sentence ψ such that for any model M,*

$$M \models \phi \iff M \models \psi;$$

and vice versa.

Proof (Idea):

- ESO $\Longrightarrow$ FO(=(...)):
  E.g., $M \models \exists f \forall \vec{x} \alpha(\vec{x}, f(\vec{x}_i)) \iff M \models \forall \vec{x} \exists y (=(\vec{x}_i, y) \wedge \alpha(\vec{x}, y))$.
- FO(=(...)) $\Longrightarrow$ ESO(R): Observation: A team X over the domain $\{v_1, \ldots, v_n\}$ induces an n-ary relation $rel(X) = \{s(\vec{v}) \mid s \in X\}$:

	v_1	v_2	$\ldots$	v_n
s_1	a_{11}	a_{12}	$\ldots$	a_{1n}
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Corollary. The classes of finite structures definable in FO(=(...)) and FO($\perp$) are exactly the ones recognized in NP. (follows from Fagin 1973)

Partial axiomatization

Theorem ( )

$$\Gamma \models \phi \iff \Gamma \vdash \phi$$

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Theorem (Kontinen, Väänänen 2013 & Hannula 2015)

There are (sound) systems of natural deduction for FO$(=(...))$ *and* FO$(\perp)$ *such that*

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for any set Γ *of sentences and first-order sentence* α.

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There are (sound) systems of natural deduction for FO(=(...)) *and* FO($\perp$) *such that*

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Def. A formula θ in L is called negatable if there exists a formula η in L s.t. $\eta \equiv \dot{\sim}\theta$, where the weak classical negation $\dot{\sim}$ is defined as

$$M \models_X \dot{\sim}\phi \iff X = \emptyset \text{ or } M \not\models_X \phi.$$

- Remark: Neither FO(=(...)) nor FO($\perp$) is closed under $\dot{\sim}$, since FO($\perp$) $\equiv$ FO(=(...)) $\equiv$ ESO.
- First-order formulas α are negatable in FO(=(...)) and FO($\perp$).
- Thm. (Y. 2016) For any formula α in FO($\perp$), $\dot{\sim}\alpha$ exists in FO($\perp$) iff the ESO-translation χ_α of α is equiv. to a first-order formula.
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Examples:

- Dependence and independence atoms are all negatable in FO($\perp$).
  $\rightsquigarrow$ Armstrong's axioms for functional dependencies are derivable.
  $\rightsquigarrow$ Some facts concerning independence notions in quantum theory are derivable. (Abramsky, Puljujärvi, Väänänen 2021)
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- If $\Gamma \models \perp$, then $\Gamma \vdash \perp$.
  $\rightsquigarrow$ Arrow's Impossibility Theorem can be formalized in FO($\perp$) as $\Gamma_{\text{Arrow}}, \dot{\sim}\phi_{\text{dictator}} \models \perp$ or $\Gamma_{\text{Arrow}} \models \phi_{\text{dictator}}$, and it is derivable in the system of FO($\perp$), i.e., $\Gamma_{\text{Arrow}} \vdash \phi_{\text{dictator}}$. (Pacuit, Y. 2016)

Weaker (and axiomatizable) team-based logics

- (Kontinen, Y. 2020) A variant of dependence logic with weaker quantifiers $\forall^1, \exists^1$ and global disjunction $\mathbin{\rotatebox[origin=c]{0}{$\vee$}}\!\!\!\vee$
- (Lück 2018) First-order logic with the (strong) classical negation $\sim$
- (Baltag, van Benthem 2021) Dependence logic with a “local” version of functional dependence

Local v.s. global disjunction

Local disjunction:

- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ s.t. $M \models_Y \phi$ & $M \models_Z \psi$.

x	y	z		
3	4	5	α	
2	3	0	α	$\alpha \vee \beta$
1	2	3	α, β	
0	1	0	β	

Global disjunction:

- $M \models_X \phi \mathbin{\vee\!\!\!\vee} \psi$ iff $M \models_X \phi$ or $M \models_X \psi$

Fact: $\phi \mathbin{\vee\!\!\!\vee} \psi \equiv \exists xy\big(\,{=}(x) \wedge {=}(y) \wedge \big((\phi \wedge x = y) \vee (\psi \wedge x \neq y)\big)\big)$

- $M \models_X \phi \to \psi$ iff for all $Y \subseteq X$, $M \models_Y \phi$ implies $M \models_Y \psi$.

— introduced by Abramsky & Väänänen (2009)

Properties:

- If ϕ and ψ are downward closed, so is $\phi \to \psi$.
- $=\!(x_1 \ldots x_n, y) \equiv =\!(x_1) \wedge \ldots \ =\!(x_n) \to =\!(y)$
- Thm (Y. 2013). FO($=\!(\ldots), \to$) $\equiv$ full second-order logic

Propositional dependence logic

Propositional dependence atoms and propositional teams

A first-order team:
a set of assignments $s : V \to M$
with $V \subseteq \mathrm{Var}$

	x	y	z
s_1	a	c	b
s_2	a	c	c
s_3	b	d	d
s_4	b	d	c

$$M \models_X =\!(\vec{x}, \vec{y})$$

"Energy is determined by mass"
(via the function $e = mc^2$)

$$=\!(\vec{p}, \vec{q})$$

"Whether I will take my umbrella
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A propositional team:
a set of valuations / possible worlds $v : V \to \{0, 1\}$ with $V \subseteq \mathrm{Prop}$

	p	q	r
v_1	1	1	1
v_2	1	1	0
v_3	0	0	0
v_4	0	0	1

$${=}(\vec{p}, \vec{q})$$

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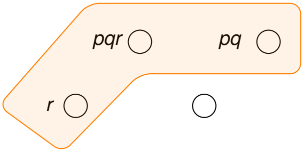

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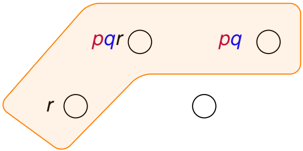

$$X \models =(\vec{p}, \vec{q}) \quad \text{iff} \quad \text{for all } v, u \in X$$
$$v(\vec{p}) = u(\vec{p}) \Longrightarrow v(\vec{q}) = u(\vec{q})$$

"Whether I will take my umbrella depends on whether it is raining."

- $X \models =\!(\vec{p})$ iff for all $v, u \in X$: $v(\vec{p}) = u(\vec{p})$.

$\ldots$	p	$\ldots$
$\ldots$	1	$\ldots$
$\ldots$	1	$\ldots$
$\ldots$	1	$\ldots$
$\ldots$	1	$\ldots$

or

$\ldots$	p	$\ldots$
$\ldots$	0	$\ldots$
$\ldots$	0	$\ldots$
$\ldots$	0	$\ldots$
$\ldots$	0	$\ldots$

Fact: $=\!(p) \equiv p \mathbin{\mathbb{V}} \neg p$

Constancy atoms revisited

- $X \models {=}(\vec{p})$ iff for all $v, u \in X$: $v(\vec{p}) = u(\vec{p})$.

...	p	...
...	1	...
...	1	...
...	1	...
...	1	...

or

...	p	...
...	0	...
...	0	...
...	0	...
...	0	...

Fact: ${=}(p) \equiv p \mathbin{\mathbb{V}} \neg p$

${=}(p_1 \ldots p_n, q) \equiv {=}(p_1) \wedge \cdots \wedge {=}(p_n) \to {=}(q)$

- $X \models =\!(\vec{p})$ iff for all $v, u \in X$: $v(\vec{p}) = u(\vec{p})$.

...	p	...
...	1	...
...	1	...
...	1	...
...	1	...

or

...	p	...
...	0	...
...	0	...
...	0	...
...	0	...

Fact: $=\!(p) \equiv p \vee\!\!\!\vee \neg p \equiv ?p$

$=\!(p_1 \dots p_n, q) \equiv =\!(p_1) \wedge \cdots \wedge =\!(p_n) \to =\!(q)$

Remark: Team semantics was adopted independently also in inquisitive semantics (Ciardelli and Roelofsen 2011) to model questions in natural language.

Language of standard logic: $\alpha ::= p \mid \bot \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \to \alpha$

Language of team-based logic (tCPC):

$\phi ::= p \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \phi \mathbin{\mathbb{V}} \phi \mid {=}(\vec{p}, \vec{q}) \qquad \neg\phi := \phi \to \bot$

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Team semantics: Let $t \subseteq 2^{\text{Prop}}$ be a team, i.e., a set of possible worlds.

- $t \models p$ iff $v(p) = 1$ for all $v \in t$
- $t \models \bot$ iff $t = \emptyset$
- $t \models \phi \wedge \psi$ iff $t \models \phi$ and $t \models \psi$
- $t \models \phi \mathbin{\mathbb{V}} \psi$ iff $t \models \phi$ or $t \models \psi$
- $t \models \phi \vee \psi$ iff $\exists s, r \subseteq t$ s.t. $t = s \cup r$, $s \models \phi$ and $r \models \psi$
- $t \models \phi \to \psi$ iff $\forall s \subseteq t$: $s \models \phi$ implies $s \models \psi$
- $t \models \neg\phi$ iff $t \models \phi \to \bot$ iff $\{v\} \not\models \phi$ for all $v \in t$

Empty team property: $\emptyset \models \phi$ for all ϕ

Downward Closure: If $s \subseteq t \models \phi$, then $s \models \phi$.

For any standard formula α (i.e., formula of the standard logic),

Union closure: $t \models \alpha$ and $s \models \alpha \Longrightarrow t \cup s \models \alpha$

Flatness: $t \models \alpha \iff \forall v \in t : \{v\} \models \alpha \iff \forall v \in t : v \models \alpha$

Propositional team-based logic

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Language of team-based logic (tCPC):

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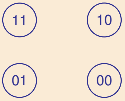
Union closure: $t \models \alpha$ and $s \models \alpha \Longrightarrow t \cup s \models \alpha$

Flatness: $t \models \alpha \iff \forall v \in t : \{v\} \models \alpha \iff \forall v \in t : v \models \alpha$

The powerset model

Fix a finite set $\mathsf{Prop}_n = \{p_1, \ldots, p_n\}$ of propositional variables.



- $t \models \phi$ iff $M^\circ, t \Vdash \phi$; & team implication = intuitionistic implication:
  - $t \models \phi \to \psi$ iff for all $s \in \wp(W)$ with $t \supseteq s$, $s \models \phi$ implies $s \models \psi$
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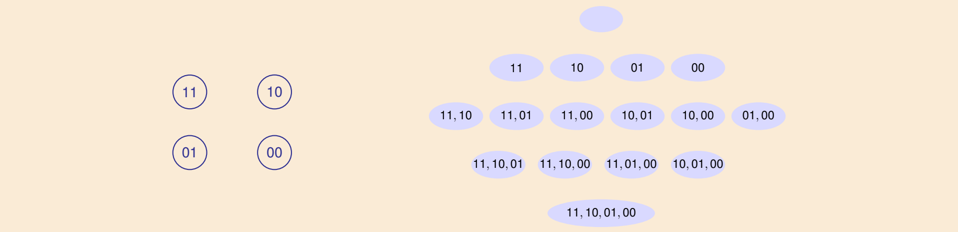

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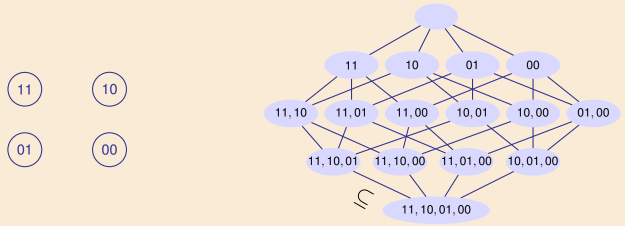

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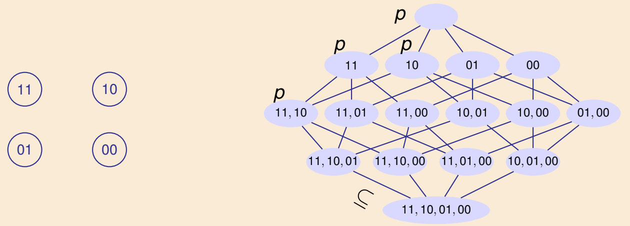

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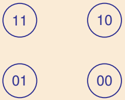

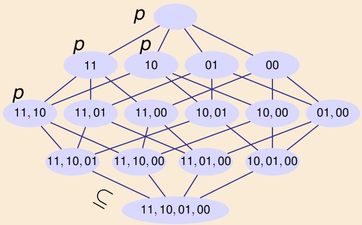

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Disjunctive normal form

- Prop (disjunctive normal form). For any formula ϕ, we have that
$$\phi \equiv \mathbb{V}_{i \in I} \alpha_i,$$
for some standard (i.e., $\vee$-free) formulas α_i.
Pf. By $(\alpha \to \phi \mathbb{V} \psi) \to (\alpha \to \phi) \mathbb{V} (\alpha \to \psi)$ (Split axiom). □

- Disjunction property: If $\models \phi \mathbb{V} \psi$, then $\models \phi$ or $\models \psi$

The sound and complete Hilbert system tCPC consists of the axioms:

- All IPC axioms for the language $[\bot, \wedge, \mathbin{\vee\mkern-11mu\vee}, \to]$, i.e.,
  - $\vdots$
  - $\phi \to (\phi \mathbin{\vee\mkern-11mu\vee} \psi)$, $\psi \to (\phi \mathbin{\vee\mkern-11mu\vee} \psi)$
  - $(\phi \to \chi) \to ((\psi \to \chi) \to ((\phi \mathbin{\vee\mkern-11mu\vee} \psi) \to \chi))$
  - $\bot \to \phi$
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- $\phi \vee (\psi \mathbin{\vee\mkern-11mu\vee} \chi) \to (\phi \vee \psi) \mathbin{\vee\mkern-11mu\vee} (\phi \vee \chi)$
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- $(\phi \to \alpha) \to ((\psi \to \alpha) \to (\phi \vee \psi \to \alpha))$
- $\phi \vee \psi \to \psi \vee \phi$
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- $\neg\neg\alpha \to \alpha$ No uniform substitution!

and the Modus Ponens rule. (Ciardelli, Roelfsen 2011), (Y., Väänänen 2016)

- (Ciardelli, Iemhoff, Y. 2020): tIPC = tCPC $\ominus$ $\neg\neg\alpha \to \alpha$ is complete for team semantics over intuitionistic Kripke models, where a team is a set of possible worlds in an intuitionistic Kripke model.
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Axiomatization

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  - $\bot \to \phi$
- $(\alpha \to \phi \Vee \psi) \to (\alpha \to \phi) \Vee (\alpha \to \psi)$ (Split)
- $\phi \vee (\psi \Vee \chi) \to (\phi \vee \psi) \Vee (\phi \vee \chi)$
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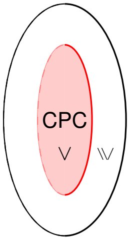
- Given an intermediate logic $\mathsf{L} = \mathsf{IPC} \oplus \alpha_1 \oplus \cdots \oplus \alpha_n$ with each $\alpha_i \in [\bot, \wedge, \vee, \to]$, define $\mathsf{tL} = \mathsf{tIPC} \oplus \alpha_1 \oplus \cdots \oplus \alpha_n$.

  (Quadrellaro 2021), cf. (Punčochář 2021)

  Thm. For any L that is complete w.r.t. a class $\mathsf{F_L}$ of frames, if L has the disjunction property or is canonical, then tL is complete w.r.t. $\mathsf{F_L}$ too.

  (Bezhanishvili , Y. 2022)



$\neg\neg\alpha \to \alpha$

- tCPC = tIPC $\oplus$ $\neg\neg\alpha \to \alpha$.
- Given L $=$ IPC $\oplus\ \Delta$, tL $=$ tIPC $\oplus\ \Delta$.
- Conservativity: For any set $\Delta \cup \{\alpha\}$ of standard formulas,

$$\Delta \vdash_{\text{tIPC}} \alpha \iff \Delta \vdash_{\text{IPC}} \alpha$$

- $\phi \equiv \mathbb{V}_{i \in I}\, \alpha_i$ (by Split axiom $(\alpha \to \phi \mathbin{\mathbb{V}} \psi) \to (\alpha \to \phi) \mathbin{\mathbb{V}} (\alpha \to \psi)$)
- Glivenko-type theorem (Ciardelli, Iemhoff, Y. 2020):

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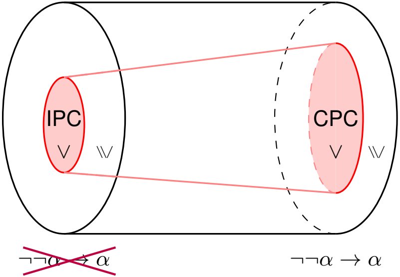
(Recall: Glivenko's theorem : $\vdash_{\text{CPC}} \alpha \iff \vdash_{\text{IPC}} \neg\neg\alpha$.)



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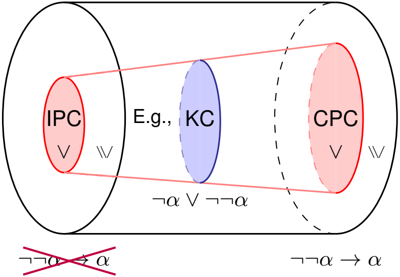

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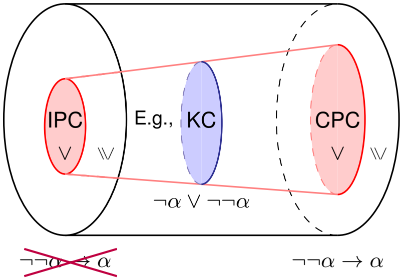

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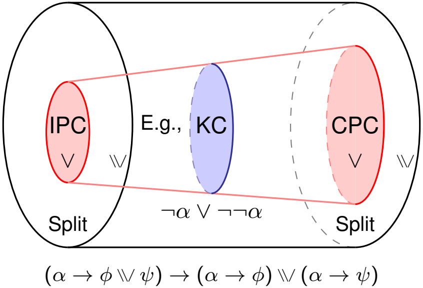
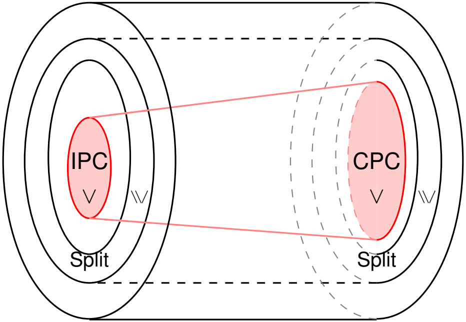
Changing the team layer? (work in progress with N. Bezhanishvili)





Replace the Split axiom $(\alpha \to \phi \mathbin{\mathbb{V}} \psi) \to (\alpha \to \phi) \mathbin{\mathbb{V}} (\alpha \to \psi)$ by other axioms?

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The powerset model revisited

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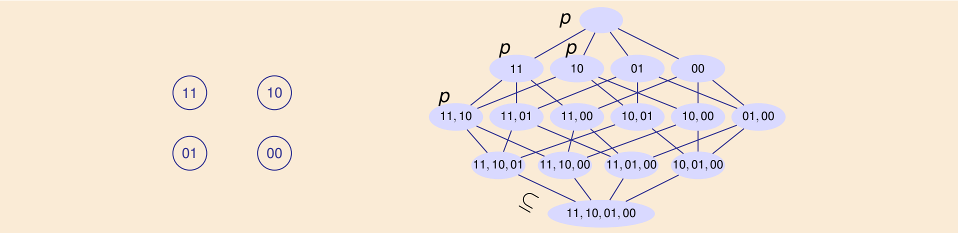

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- The structure $(\wp(W), \cup, \emptyset)$ forms a bounded join-semilattice with $\subseteq$ the induced partial orderd (i.e., $s \subseteq t$ iff $s \cup t = t$).

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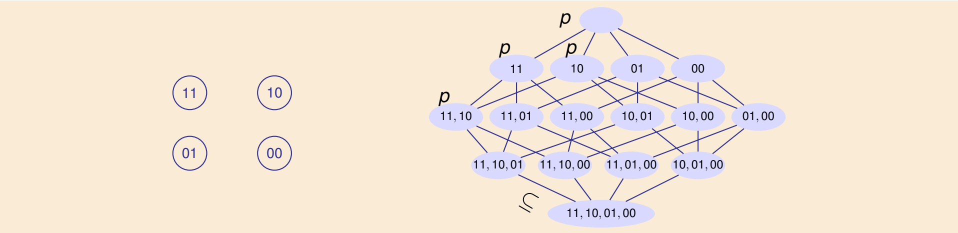

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General team models

Definition

A general team model is a tuple $M = (\wp(W), \emptyset, \Cup, \succcurlyeq, V)$, *where*

- $(\wp(W), \Cup, \emptyset)$ *is a bounded join semi-lattice with* $\preccurlyeq$ *the induced partial order (i.e.,* $s \preccurlyeq t$ *iff* $s \Cup t = t$*);*
- $V : \mathrm{Prop} \to \wp(\wp(W))$ *is such that* $V(p)$ *is a principle upset w.r.t.* $\succcurlyeq$, *i.e.,* $V(p) = t^{\uparrow} = \{s \in \wp(W) \mid t \succcurlyeq s\}$ *for some* $t \in \wp(W)$.

cf. (Punčochář 2017), (Dmitrieva 2021)

Definition

A *general team model* *is a tuple* $M=(A,0,\Cup,\succcurlyeq,V)$, *where*

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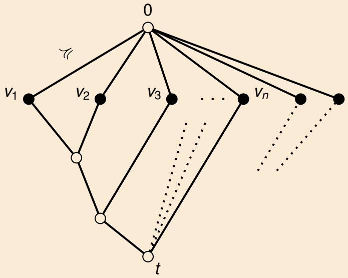
cf. (Punčochář 2017), (Dmitrieva 2021)



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General team models

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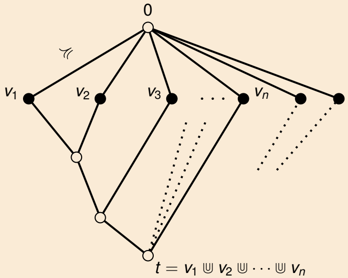

Def. A bounded semi-lattice $(A, \Cup, 0)$ is called atomistic if every non-zero element $t \in A$ is a finite join of atoms, i.e., $t = v_1 \Cup \cdots \Cup v_n$ for some atoms $v_1, \ldots, v_n \in A$.

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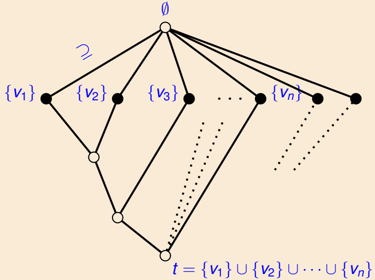

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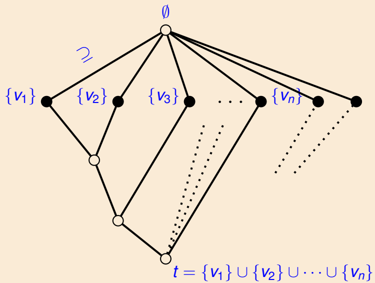
Standard team semantics: Given the powerset model $M = (\wp(2^{\mathsf{Prop}_n}), \emptyset, \cup, \supseteq, V)$, and a team $t \subseteq 2^{\mathsf{Prop}_n}$,

- $M, t \models =\!(p, q)$ iff for all $\{u\}, \{v\} \subseteq t$: $u \models p \Leftrightarrow v \models p$ implies $u \models q \Leftrightarrow v \models q$



- Amstrong's axioms for functional dependence still hold.

Functional dependence

Generalized team semantics: Given a general team model $M = (A, 0, \Cup, \succcurlyeq, V)$ and a team $t \in A$,

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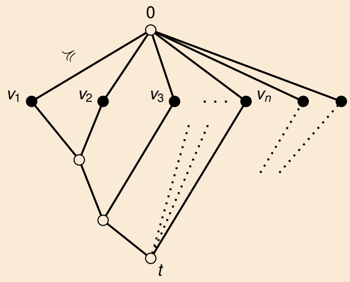

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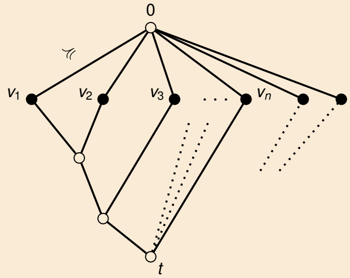

- Amstrong's axioms for functional dependence still hold.

Standard team semantics:

Given the powerset model $M = (\wp(W), \emptyset, \cup, \supseteq, V)$ with $W = 2^{\mathsf{Prop}}$, and a team $t \subseteq W$.

- $M, t \models \bot$ iff $t = \emptyset$
- $M, t \models \phi \vee \psi$ iff there are $s, r \in \wp(W)$ such that $t \subseteq s \cup r$, $M, s \models \phi$ and $M, r \models \psi$
- $M, t \models \phi \to \psi$ iff for all $s \in \wp(W)$ with $t \supseteq s$, $M, s \models \phi$ implies $M, s \models \psi$

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Persistency / Downward closure: If $M, t \models \phi$ and $t \succcurlyeq s$, then $M, s \models \phi$.

Empty team property: $M, 0 \models \phi$ for all ϕ.

The logic of the generalized semantics

Under the generalized semantics, all tCPC axioms hold except for

1. $\neg\neg\alpha \to \alpha$ (Double negation elimination)
2. $(\alpha \to \phi \mathbin{\mathbb{V}} \psi) \to (\alpha \to \phi) \mathbin{\mathbb{V}} (\alpha \to \psi)$ (Split)
3. $(\phi \to \chi) \to (\phi \vee \psi \to \chi \vee \psi)$ (Monotonicity)
4. $(\phi \to \alpha) \to ((\psi \to \alpha) \to (\phi \vee \psi \to \alpha))$

where $\alpha \in [\bot, \wedge, \vee, \to]$ is a standard formula

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Standard formulas are not any more closed under joins, where:
Join Closure Property: $M, t \models \alpha$ and $M, s \models \alpha \Longrightarrow M, t \Cup s \models \alpha$
(If $\Cup = \cup$, join closure is the union closure property)

The logic of the generalized semantics, and distributivity

Under the generalized semantics, all tCPC axioms hold except for

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Fact: Split axiom holds over distributive frames F, i.e., semi-lattices $F = (A, 0, \Cup, \succcurlyeq)$ s.t.

$$t \preccurlyeq r \Cup s \Longrightarrow \exists r', s' \in A : r' \preccurlyeq r,\ s' \preccurlyeq s,\ \text{ and }\ t = r' \Cup s'.$$

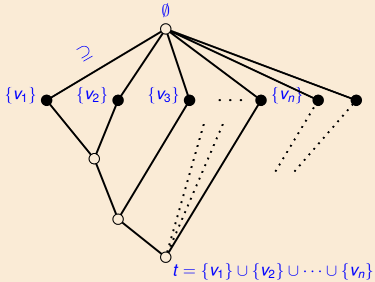
Thm. For finite frames F, we have that F satisfies join closure over standard formulas and validates ❸ and ❹ iff F is distributive.

Flatness of standard formulas

Over standard team semantics:

Def. A formula ϕ is said to be flat, if

$$t \models \phi \iff \text{for all } \{v\} \subseteq t: \ \{v\} \models \phi$$

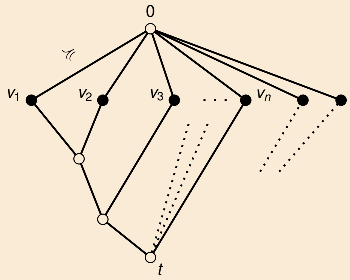


Prop. Standard formulas $\alpha \in [\bot, \wedge, \vee, \to]$ are flat.

Over generalized team semantics:

Def. A formula ϕ is said to be flat over M, if

$$M, t \models \phi \iff \text{for all atoms } v \preccurlyeq t: \; M, v \models \phi$$



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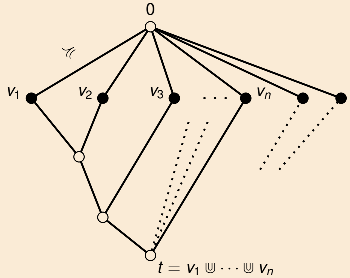

~~Prop. Standard formulas $\alpha \in [\bot, \wedge, \vee, \to]$ are flat.~~

Thm. Let F be a finite frame. Then,

F is atomistic iff all standard formulas α are flat over F

iff $F \models \neg\neg\alpha \to \alpha$ for all standard formulas α.