A pronilpotent look at maximal subgroups of free profinite monoids

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UNIVERSIDADE DO PORTO

Part 1

Setting the stage

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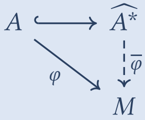

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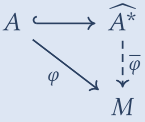

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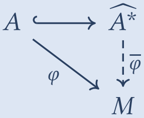

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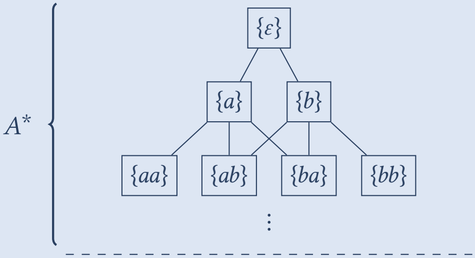
- Elements of $\widehat{A^*}$ are called **pseudowords**.

$\mathcal{J}$-classes in $\widehat{A^*}$



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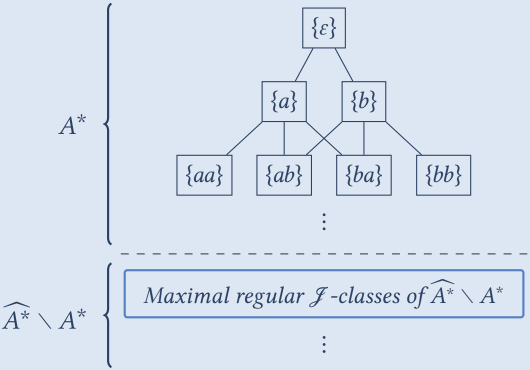

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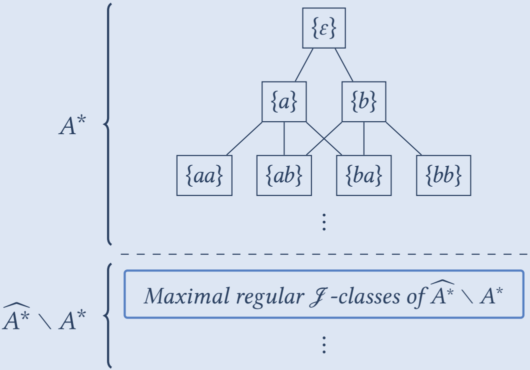

Main goal: study the maximal subgroups inside the regular $\mathcal{J}$-classes found in the "top layer" of $\widehat{A^*} \setminus A^*$.

Almeida's correspondence

Theorem (Almeida, 2007)

The following is a bijection between *uniformly recurrent languages* $L \subseteq A^*$ and *maximal regular* $\mathcal{J}$*-classes* $J \subseteq \widehat{A^*} \setminus A^*$:

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Rhodes and Steinberg, 2008: $G(L)$ is a projective profinite group.

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A uniformly recurrent language $L \subseteq A^*$ gives a regular $\mathscr{J}$-class $J(L) \subseteq \widehat{A^*}$.

The $\mathscr{J}$-class $J(L)$ gives a profinite group $G(L)$, the *Schützenberger group*.

The group $G(L)$ is a maximal subgroup of $\widehat{A^*}$ and a projective profinite group.

Part 2

Pronilpotent quotients of projective profinite groups

Profinite groups

- A **profinite group** is a compact group G whose identity 1_G has a neighbourhood basis of clopen normal subgroups $N \trianglelefteq G$ such that

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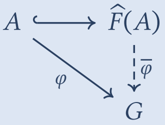

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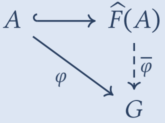

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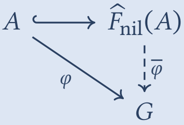
Profinite groups

- A **pronilpotent group** is a compact group G whose identity 1_G has a neighbourhood basis of clopen normal subgroups $N \trianglelefteq G$ such that

$$G/N \text{ is a finite nilpotent group.}$$

- The category of pronilpotent groups has free objects $\widehat{F}_{\mathrm{nil}}(A)$.



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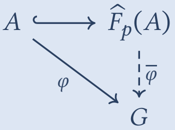

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- A **pro-H** group is a compact group G whose identity 1_G has a neighbourhood basis of clopen normal subgroups $N \trianglelefteq G$ such that

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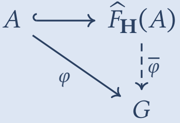
- The category of pro-**H** groups has free objects $\widehat{F}_{\mathbf{H}}(A)$.



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- $\widehat{F}_{\mathbf{H}}(A)$ can be seen as a completion of the free group $F(A)$.
- We assume that **H** is a **pseudovariety** of finite groups (closed under finite direct products, quotients and subgroups).

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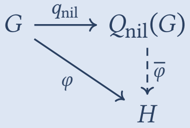

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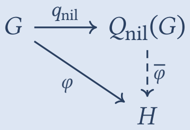

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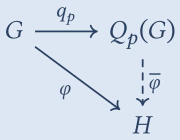

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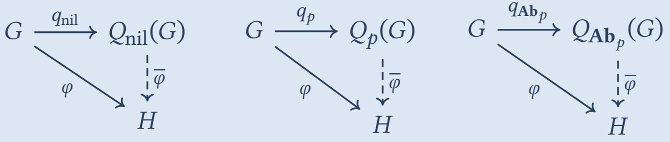

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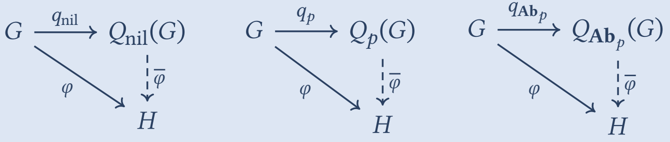

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- Also: $Q_{\mathbf{Ab}_p}(G)$ is the *Frattini quotient* of $Q_p(G)$.
- There is a natural isomorphism $Q_{\mathrm{nil}} \cong \prod_p Q_p$.

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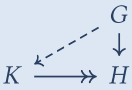

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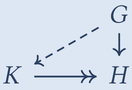

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Proposition

If G is a projective profinite group, then

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Almeida and Costa, 2013: in some cases, Schützenberger groups of uniformly recurrent languages are ω-presented.

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Theorem

If $\varphi : F(A) \to F(A)$ defines an ω-presentation of G, then

$$\mathrm{d}_p(G) = \deg(\chi^*_{p,\varphi}).$$

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- For $a \in A$, extend the delta function $\delta_a : A \to \{0, 1\}$ to a homomorphism $|-|_a : F(A) \to \mathbb{Z}$ ("counting occurrences" of a).
- Define the **composition matrix** of $\varphi : F(A) \to F(A)$ by

$$M_\varphi(a, b) = |\varphi(b)|_a, \quad a, b \in A.$$

- Let $\chi_{p,\varphi}$ be the characteristic polynomial of M_φ over $\mathbb{Z}/p\mathbb{Z}$.
- The **reciprocal** of a degree n polynomial ξ is $\xi^*(x) = x^n \xi(x^{-1})$.

Theorem

If $\varphi : F(A) \to F(A)$ defines an ω-presentation of G, then

$$\mathrm{d}_p(G) = \deg(\chi^*_{p,\varphi}).$$

In particular, $Q_{\mathrm{nil}}(G) \cong \prod_p \widehat{F}_p(\deg(\chi^*_{p,\varphi}))$.

Corollaries

Suppose that $G \cong \langle A \mid \hat{\varphi}(a)a^{-1} : a \in A \rangle$ is ω-presented.

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Corollary 3

If there are primes p, q such that $0 < \deg(\chi^*_{p,\varphi}) < \deg(\chi^*_{q,\varphi})$, then G is not relatively free.

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Recap

When G is ω-presented by an endomorphism φ,

$Q_{\text{nil}}(G)$ is completely determined by the prime-indexed sequence $(\deg(\chi^*_{p,\varphi}))_p$.

Part 3

Schützenberger groups of primitive substitutions

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- Recall *Almeida's theorem*: to each uniformly recurrent language $L \subseteq A^*$ corresponds a maximal subgroup $G(L) \subseteq \widehat{A^*}$.
- For short, we write $G(\varphi)$ instead of $G(L(\varphi))$.

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A uniformly recurrent language $L \subseteq A^*$ is **aperiodic** if $L \cap \{w^n : n \geq 0\}$ is finite for all $w \in L$.

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- *Durand, 2012*: There is an algorithm which takes as input a primitive substitution, and outputs a return substitution.
- *But* the algorithm can be costly and unpredictable.

Durand's lemma

Lemma (Durand, 1998)

Let φ be a primitive aperiodic substitution with a return substitution ψ.
For some $n > 0$ and some products of cyclotomic polynomials ξ_1, ξ_2,

$$\xi_1 \chi^*_{\varphi^n} = \pm \xi_2 \chi^*_{\psi}.$$

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Therefore: $(\deg(\chi^*_{p,\varphi}))_p$ determine $Q_{\mathrm{nil}}(G(\varphi))$ *up to a free pronilpotent factor.*

This is true even though φ may not define an ω-presentation of $G(\varphi)$.

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Corollary 3

If there are primes p, q, r such that $\deg(\chi^*_{p,\varphi}) < \deg(\chi^*_{q,\varphi}) < \deg(\chi^*_{r,\varphi})$, then $G(\varphi)$ is not relatively free.

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Thank you for your attention!