Sums of Kripke frames and locally finite modal logics

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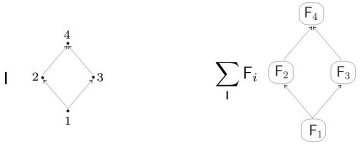
Unimodal case:
frame of indices $\mathsf{I} = (I, S)$;
frames-summands $\mathsf{F}_i = (W_i, R_i)$, i in I.



For classes $\mathcal{I}$, $\mathcal{F}$ of frames, $\sum_{\mathcal{I}} \mathcal{F}$ is the class of all sums $\sum_{i \in \mathsf{I}} \mathsf{F}_i$ such that $\mathsf{I} \in \mathcal{I}$ and $\mathsf{F}_i \in \mathcal{F}$ for every i in I.

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Idea: To study the modal logic of a class of sums via logics of summands/indices.

This is not a new approach:
In classical model theory, "*composition theorems*" reduce the theory (FO, MSO) of a compound structure to theories of its components ([Feferman–Vaught 1959], [Shelah 1975], [Gurevich 1979], ...)

General observation:
In many cases, the modal satisfiability problem on sums can be reduced to the modal satisfiability problem on summands.This gives transfer results for

- finite model property and decidability,
- computational complexity,
- local finiteness.

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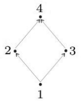

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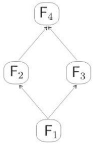

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Many important normal modal logics can by characterized as logics of *sums* of relational structures.

[Beklemishev 2007] Iterated sums over Noetherian orders are models for Japaridze's polymodal provability logic *GLP*.

[Balbiani 2009; Balbiani and Mikulás 2013; Balbiani and Fernández-Duque 2016]: Lexicographic products of modal logics

[Babenyshev and Rybakov 2010] Refinement of modal logics

[Sh 2008; 2020] *GLP* is decidable in PSpace. In general, the sum operation over Noetherian orders preserves "good" computational properties (satisfiability is sums is polynomial space Turing reducible to summands).

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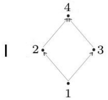

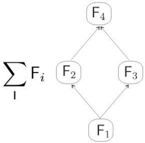

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For simplicity of notation, results below are formulated for the unimodal case. They work for the polymodal case as well.

Theorem. Let $\mathcal{I}$, $\mathcal{F}$, $\mathcal{G}$ be classes of frames.

- Corollary of [Babenyshev and Rybakov 2010]: If $\mathrm{Log}\,\mathcal{I}$ and $\mathrm{Log}\,\mathcal{F}$ admit filtration, then $\mathrm{Log}\sum_{\mathcal{I}}\mathcal{F}$ admits filtration.
- [Sh 2018] Put $\mathcal{F}\equiv\mathcal{G}$ iff $\mathcal{F}$ and $\mathcal{G}$ have the same modal logic in the language enriched with the universal modality. We have for any $\mathcal{I}$:
  If $\mathcal{F}\equiv\mathcal{G}$, then $\sum_{\mathcal{I}}\mathcal{F}\equiv\sum_{\mathcal{I}}\mathcal{G}$.
- [Sh 2018] If $\mathcal{I}$ is a class of Noetherian orders that contains all finite trees, then

  $$\mathrm{Log}\sum_{\mathcal{I}}\mathcal{F}=\mathrm{Log}\sum_{\text{finite trees}}\mathcal{F}$$

  In particular, if $\mathrm{Log}\,\mathcal{F}^{\forall}$ has the FMP, then so does $\mathrm{Log}\sum_{\mathcal{I}}\mathcal{F}$: it is complete w.r.t.

  $$\sum_{\text{finite trees}}\{\text{finite frames of }\mathrm{Log}\,\mathcal{F}\}.$$

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Informally, filtration is a method of collapsing an infinite model into a finite one while preserving the truth value of a given formula. It is widely used as a tool for establishing the finite model property and decidability of modal logics.

A logic L *admits filtration* iff any L-model can be "filtrated" into a finite L-model.

$$L \text{ admits filtration} \quad \begin{smallmatrix}\Rightarrow \\ \not\Leftarrow\end{smallmatrix} \quad L \text{ has the fmp.}$$

Many standard modal logics admit filtration.

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Universal modality on a set W is interpreted by the relation $W \times W$.

Enriching modal language with universal modality does not necessarily preserve the fmp/decidability [Wolter 94; Spaan 1993].

Fortunately, in many cases (for example, for logics that admit filtration or for logics of transitive relations) it does [Goranko and Passi 1991; Spaan 1996].

[Simon and Gill 1977]
Polynomial space Turing reductions:
For problems A and B, $A \leq_{\mathrm{T}}^{\mathrm{PSpace}} B$ iff there exists a polynomial space bounded oracle deterministic machine M with oracle B that recognizes A.

$$A \leq_{\mathrm{T}}^{\mathrm{PSpace}} B \in \mathrm{PSpace} \;\Rightarrow\; A \in \mathrm{PSpace}$$

Theorem [Sh 2020] Let $\mathcal{F}$ be a class of frames, $\mathcal{I}$ a class of Noetherian orders containing all finite trees. Then:

- $\mathrm{Sat} \sum_{\mathcal{I}} \mathcal{F} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{Sat}\, \mathcal{F}^{\forall}$.
- If also $\mathcal{I}$ is closed under finite disjoint unions, then $\mathrm{Sat}\, (\sum_{\mathcal{I}} \mathcal{F})^{\forall} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{Sat}\, \mathcal{F}^{\forall}$.

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in many cases (e.g., when $\mathcal{F}$ is the class of frames of a transitive logic); hence:

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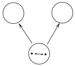
Example. The logic of preorders $\mathrm{S4}$

[McKinsey 1941] $\mathrm{S4}$ has the FMP, so is decidable.

[Ladner 1977] $\mathrm{S4} \in \mathrm{PSpace}$.

Complexity via sums: Clusters are frames of form

$$(C, C \times C).$$



Every preorder is a sum $\sum_{\text{partial order}}$ (clusters). Hence $\mathrm{S4}$ is the logic of the class

$$\sum_{\text{finite posets}} \text{clusters}.$$

Thus:

$$\mathrm{Sat(preorders)} \le_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{Sat(clusters)}$$

The satisfiability on clusters is (trivially) in NP, so is in PSpace.

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Example. The logic of weakly transitive relations $\mathrm{wK4}$

R is *weakly transitive* iff

$$xRyRz \;\Rightarrow\; xRz \vee x = z$$

[Esakia 2001]
1. $\mathrm{wK4}$ is the logic of all topological spaces, where $\Diamond$ is the topological derivative.
2. $\mathrm{wK4}$ has the FMP and decidable.

Corollary. $\mathrm{wK4} \in \mathrm{PSpace}$.

Proof. Because of the FMP, $\mathrm{wK4}$ is the logic of

$$\sum_{\text{finite PO}} \mathcal{C},$$

where
(W, R) is in $\mathcal{C}$ iff R contains the difference relation:

$$x \neq y \Rightarrow xRy.$$

A simple fact: $\mathrm{Sat}\, \mathcal{C}$ is in NP. □

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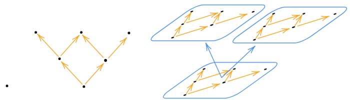
Example. Polymodal Provability Logic *GLP* [Japaridze 1986].

GLP is an important system in proof theory. It axiomatizes so called *graded provability algebras* (Lindenbaum boolean algebras of formal theories like PA enriched by provability operators $[0], [1], [2]$ of different strength).
GLP is Kripke-incomplete.

[Beklemishev 2007] *GLP* is polynomial-time reducible to the logic of iterated sums over Noetherian orders:



Corollary. *GLP* $\in \mathrm{PSpace}$.
Proof (sketch).

$$\mathrm{Sat}(\{\textit{singleton}\}) \in \mathrm{NP}$$

□

Local finiteness

The *algebra* $\mathrm{Alg}(F)$ *of a frame* $F = (X, (R_a)_{a \in \mathrm{A}})$ is the powerset algebra of X endowed with

$$\Diamond_a : \mathcal{P}(X) \to \mathcal{P}(X),$$

where for $Y \subseteq X$, $\Diamond_a(Y) = R_a^{-1}[Y]$.

$\mathrm{Log}(\mathsf{F})$ is LF $\underset{\not\Leftarrow}{\Rightarrow}$ $\mathrm{Alg}(\mathsf{F})$ is LF $\underset{\not\Leftarrow}{\Rightarrow}$ $\mathrm{Log}(\mathsf{F})$ has the FMP.

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Main lemma (2022) Let $\mathrm{A} < \omega$ be the alphabet of modal operators. Let $(\mathsf{F}_i)_{i \in I}$ be a family of A-frames, $\mathsf{I} = (I, (S_a)_{\mathrm{A}})$ be an A-frame with all S_a irreflexive.

- If the algebras $\mathrm{Alg}(\bigsqcup_I \mathsf{F}_i)$ and $\mathrm{Alg}(\mathsf{I})$ are locally finite, then $\mathrm{Alg}(\sum_{\mathsf{I}} \mathsf{F}_i)$ is locally finite.
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[Malcev, 1960s] The variety $\mathrm{Var}(\mathsf{A})$ of a finite signature is LF iff $\exists f : \omega \to \omega$ s.t. the cardinality of a subalgebra of A generated by $m < \omega$ elements is $\leq f(m)$.

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The *lexicographic sum* $\sum^{\mathrm{lex}}_{\mathsf{I}} \mathsf{F}_i$ is the $(1+\mathrm{A})$-frame $(\bigsqcup_{i \in I} W_i, S^{\mathrm{lex}}, (R_a)_{a<N})$, where

$$(i, w) S^{\mathrm{lex}} (j, u) \quad \text{iff} \quad iSj,$$
$$(i, w) R_a (j, u) \quad \text{iff} \quad i = j \ \&\ wR_{i,a}u.$$

For a class $\mathcal{F}$ of A-frames and a class $\mathcal{I}$ of 1-frames, $\sum^{\mathrm{lex}}_{\mathcal{I}} \mathcal{F}$ denotes the class of all sums $\sum^{\mathrm{lex}}_{\mathsf{I}} \mathsf{F}_i$, where $\mathsf{I} \in \mathcal{I}$ and all F_i are in $\mathcal{F}$.

Theorem (2022). If $\mathrm{Log}(\mathcal{F})$ and $\mathrm{Log}(\mathcal{I})$ are LF, then $\mathrm{Log}(\sum^{\mathrm{lex}}_{\mathcal{I}} \mathcal{F})$ is LF.

The *algebra* $\mathrm{Alg}(F)$ *of a frame* $F = (X, (R_a)_{a\in \mathrm{A}})$ is the powerset algebra of X endowed with

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Formulas of finite height (unimodal case):

$$B_0 = \bot, \quad B_{i+1} = p_{i+1} \to \Box(\lozenge p_{i+1} \vee B_i)$$

[Segerberg 1971; Maksimova 1975]
The logic of a class of transitive frames is locally finite iff it contains one of B_i's.

The non-transitive and polymodal cases are much less studied...

[Balbiani 2009] The following formulas are valid in every lexicographic sum:
$\alpha = \lozenge_1\lozenge_0 p \to \lozenge_0 p$, $\beta = \lozenge_0\lozenge_1 p \to \lozenge_0 p$,
$\gamma = \lozenge_0 p \to \Box_1\lozenge_0 p$.
Moreover, in many cases

$$\sum_{L_1}^{\mathrm{lex}} L_2 = L_1 * L_2 + \{\alpha, \beta, \gamma\},$$

where $L_1 * L_2$ denotes the fusion.

Theorem (2022). Let L_1 and L_2 be locally finite canonical unimodal logics. If the class $\mathrm{Frames}\, L_1$ is definable in first-order language without equality, then the logic

$$L_1 * L_2 + \{\alpha, \beta, \gamma\}$$

is locally finite.

Thank you!