Elementary fibrations and groupoids

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joint work with Fabio Pasquali and Giuseppe Rosolini

Topology, Algebra and Categories in Logic

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Outline

- Lawvere's doctrines as algebraic specifications of logical theories.
- Logical theories with equality as coalgebras for a comonad on theories with conjunctions.
- Lift this characterisation to non-faithful fibrations (proof-relevant theories).

J.E., F. Pasquali, G. Rosolini. Elementary doctrines as coalgebras. *J. Pure Appl. Algebra* 224, 2020.

J.E., F. Pasquali, G. Rosolini. A 2-comonad for elementary fibrations. To appear.

Doctrines

A doctrine consists of a category $\mathcal{C}$ with finite products and a functor

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{P} \mathbf{Pos}$$

F.W. Lawvere. Adjontness in foundations. *Dialectica* 1969, also in *Repr. TAC*.

B. Jacobs. *Categorical logic and type theory*. North Holland 1999.

M.E. Maietti and G. Rosolini. Quotient completion for the foundation of constructive mathematics. *Log. Univers.* 7, 2013.

Doctrines from theories

T a theory in (a fragment of) a first order multi-sorted language.

$\Rightarrow$ The syntactic doctrine $\mathsf{P}_T \colon \mathcal{C}t\chi_T^{\mathrm{op}} \longrightarrow \mathbf{Pos}$.

Doctrines from theories

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$\Rightarrow$ The syntactic doctrine $\mathrm{P}_T \colon \mathcal{C}t\chi_T^{\mathrm{op}} \longrightarrow \mathbf{Pos}$.

- Indexed on the category of contexts:

$$(y_1 : S'_1, \ldots, y_m : S'_m) \xrightarrow{(t_1,\ldots,t_m)} (x_1 : S_1, \ldots, x_n : S_n)$$

- Fibres are Lindenbaumm–Tarski algebras:

$$\mathrm{P}_T(x_1 : S_1, \ldots, x_n : S_n) := \left\{ [\alpha] \;\middle|\; \alpha \in \mathrm{WFF}(x_1 : S_1, \ldots, x_n : S_n) \right\}$$

$$\alpha \leq \beta \qquad \text{iff} \qquad x_1 : S_1, \ldots, x_n : S_n \,|\, \alpha \vdash_T \beta$$

- Reindexing is substitution of terms into formulas:

$$\mathrm{P}_T(y_1 : S'_1, \ldots, y_m : S'_m) \xleftarrow{(t_1,\ldots,t_m)^*} \mathrm{P}_T(x_1 : S_1, \ldots, x_n : S_n)$$

$$\alpha[{}^{t_1}/_{y_1}, \ldots, {}^{t_m}/_{y_m}] \longleftarrow\!\!\!\shortmid\; \alpha$$

More examples of doctrines

1. The power-set doctrine $\mathscr{P}\colon \mathcal{S}et^{\mathrm{op}} \longrightarrow \mathbf{Pos}$.

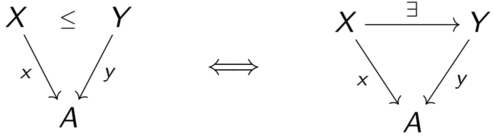
$$A \xrightarrow{f} B \quad \longmapsto \quad \mathscr{P}(B) \xrightarrow{f^{-1}} \mathscr{P}(A)$$

2. The subobjects doctrine $\mathsf{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$, for a category $\mathcal{C}$ with finite limits.

$$A \xrightarrow{f} B \quad \longmapsto \quad \mathsf{Sub}_{\mathcal{C}}(B) \xrightarrow{f^{*}} \mathsf{Sub}_{\mathcal{C}}(A)$$

3. The weak subobjects doctrine $\mathsf{wSub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$, for a category $\mathcal{C}$ with finite products and (weak) pullbacks.

$$A \xrightarrow{f} B \quad \longmapsto \quad (\mathcal{C}/B)_{\mathsf{po}} \xrightarrow{f^{*}} (\mathcal{C}/A)_{\mathsf{po}}$$



Also, for $\mathcal{A} \subseteq \mathsf{Arr}(\mathcal{C})$ stable under (weak) pullbacks, take $(\mathcal{A} \cap \mathcal{C}/A)_{\mathsf{po}}$ as the fibre over A.

Elementary doctrines

A doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is primary if

- $P(A)$ has finite meets for every A in $\mathcal{A}$, and
- $f^*\colon P(B) \longrightarrow P(A)$ preserves finite meets for every $f\colon A \longrightarrow B$.

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A primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is elementary if, for every A and I in $\mathcal{C}$, there are left adjoints

$$P(I \times A \times A) \underset{\mathrm{pr}_{1,2,2}{}^*}{\overset{\exists_{I,A}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} P(I \times A)$$

where $\mathrm{pr}_{1,2,2} := \langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle\colon I \times A \longrightarrow I \times A \times A$, which satisfy

- Beck-Chevalley, and
- Frobenius Reciprocity: $\exists_{I,A}(\alpha \wedge \mathrm{pr}^*_{1,2,2}\beta) = \exists_{I,A}(\alpha) \wedge \beta$

Elementary doctrines

Proposition

A primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is elementary iff
for every A in $\mathbf{C}$ there is $\mathrm{Eq}_A \in P(A \times A)$, and these are

Reflexive: $x : A \mid \top_A \vdash \mathrm{Eq}_A(x, x)$

Substitutive: $x_1 : A,\ x_2 : A \mid \alpha(x_1) \wedge \mathrm{Eq}_A(x_1, x_2) \vdash \alpha(x_2)$

Productive: $z : A \times B,\ z' : A \times B \ \mid$

$$\mathrm{Eq}_A(\mathrm{pr}_1(z), \mathrm{pr}_1(z')) \wedge \mathrm{Eq}_B(\mathrm{pr}_2(z), \mathrm{pr}_2(z')) \vdash \mathrm{Eq}_{A \times B}(z, z')$$

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Then

$$\exists_{I,A}(\alpha) = \mathrm{pr}^*_{1,2}\alpha \wedge \mathrm{pr}^*_{2,3}\mathrm{Eq}_A = [i : I,\ x_1 : A,\ x_2 : A \mid \alpha(x_1) \wedge \mathrm{Eq}_A(x_1, x_2)]$$

and $\exists_{I,A} \dashv \mathrm{pr}^*_{1,2,2}$ amounts to

$$i : I,\ x_1 : A,\ x_2 : A \mid \alpha(x_1) \wedge \mathrm{Eq}_A(x_1, x_2) \vdash \beta(x_1, x_2)$$

$$\text{iff} \quad i : I,\ x : A \mid \alpha(x) \vdash \beta(x, x)$$

Elementary doctrines - Examples

1. The syntactic doctrine $\mathrm{P}_T \colon \mathcal{C}tx_T^{\mathrm{op}} \longrightarrow \mathbf{Pos}$, when T is a theory in the $\top\wedge=$-fragment.

$$\mathrm{Eq}_A := \left[x_1 : A,\, x_2 : A \,\middle|\, x_1 =_A x_2\right]$$

2. The power-set doctrine $\mathscr{P} \colon \mathcal{S}et^{\mathrm{op}} \longrightarrow \mathbf{Pos}$.

$$\mathrm{Eq}_A := \left\{(x, x) \,\middle|\, x \in A\right\} \in \mathscr{P}(A \times A)$$

3. The subobjects doctrine $\mathrm{Sub}_{\mathcal{C}}$, when $\mathcal{C}$ is a category with finite limits.

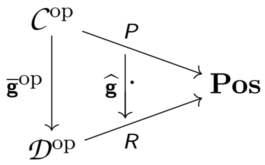
$$\mathrm{Eq}_A := [\Delta_A \colon A \hookrightarrow A \times A] \in \mathrm{Sub}_{\mathcal{C}}(A \times A)$$

   Similarly for the weak subobjects doctrine $\mathrm{wSub}_{\mathcal{C}}$.

4. The subdoctrine $A \mapsto (\mathcal{M} \cap \mathcal{C}/A)_{\mathsf{po}}$ of $\mathrm{wSub}_{\mathcal{C}}$, when $\mathcal{M}$ is the class of monos of a stable factorisation system on $\mathcal{C}$.

The 2-category $\mathbf{Doc}$ of doctrines

A morphism of doctrines $\mathbf{g}\colon P \longrightarrow R$ is a pair $\mathbf{g} = (\overline{\mathbf{g}}, \widehat{\mathbf{g}})$ where $\overline{\mathbf{g}}\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a *product-preserving* functor and



Example:

For T a first-order theory, morphisms $\mathrm{P}_T \longrightarrow \mathscr{P}$ that preserve the corresponding structure of P_T are models à la Tarski of T.

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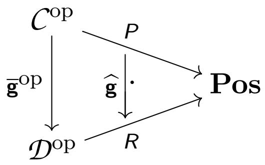


Example:
For T a first-order theory, morphisms $\mathrm{P}_T \longrightarrow \mathscr{P}$ that preserve the corresponding structure of P_T are models à la Tarski of T.

A 2-morphism $\theta\colon \mathbf{f} \Rightarrow \mathbf{g}$ is a natural transformation $\theta\colon \overline{\mathbf{f}} \stackrel{\cdot}{\longrightarrow} \overline{\mathbf{g}}$ that "preserves validity".
Example:
When T is classical and with equality, a 2-morphism $\theta\colon \mathbf{f} \Rightarrow \mathbf{g}\colon \mathrm{P}_T \longrightarrow \mathscr{P}$ is an elementary embedding.

Elementary doctrines as coalgebras

$$\mathbf{ED} \hookrightarrow \mathbf{PD}$$

Elementary doctrines as coalgebras

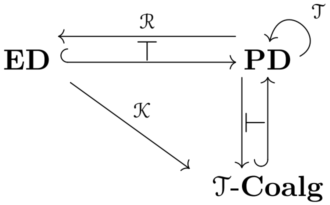
$$\mathbf{ED} \underset{\longleftarrow\;\mathcal{R}\;}{\overset{}{\hookrightarrow}} \mathbf{PD} \qquad (\mathbf{ED} \hookrightarrow \mathbf{PD} \dashv \mathcal{R})$$

Theorem[1]

The forgetful 2-functor $\mathbf{ED} \longrightarrow \mathbf{PD}$ has a right biadjoint.

[1] F. Pasquali. A co-free construction for elementary doctrines. *Appl. Categ. Structures* 23, 2015.

Elementary doctrines as coalgebras



Theorem[1]

The forgetful 2-functor $\mathbf{ED} \longrightarrow \mathbf{PD}$ has a right biadjoint.

Theorem[2]

- The 2-comonad $\mathcal{T}$ is lax-idempotent.
- The canonical comparison 2-functor $\mathcal{K}$ is a 2-isomorphism.

[1] F. Pasquali. A co-free construction for elementary doctrines. *Appl. Categ. Structures* 23, 2015.

[2] J.E., F. Pasquali, G. Rosolini. Elementary doctrines as coalgebras. *J. Pure Appl. Algebra* 224, 2020.

The underlying 2-functor of the 2-comonad $\mathcal{T}$

For $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ a primary doctrine,
the primary doctrine $\mathcal{T}(P)$ is $\mathcal{D}es_P\colon P\text{-}\mathcal{E}q\mathcal{R}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ where

$P\text{-}\mathcal{E}q\mathcal{R}$ is the category of P-equivalence relations: pairs (A, ρ) s.t.

- A in $\mathcal{C}$
- $\rho \in P(A \times A)$
- $x : A \,|\, \top_A \vdash \rho(x, x)$
- $x_1 : A,\ x_2 : A \,|\, \rho(x_1, x_2) \vdash \rho(x_2, x_1)$
- $x_1 : A,\ x_2 : A,\ x_3 : A \,|\, \rho(x_1, x_2) \wedge \rho(x_2, x_3) \vdash \rho(x_1, x_3)$

$\mathcal{D}es_P(A, \rho)$ is the sub-poset of $P(A)$ on the descent data:

$$\{\alpha \in P(A) \quad \text{s.t.} \quad x_1 : A,\ x_2 : A \,|\, \alpha(x_1) \wedge \rho(x_1, x_2) \vdash \alpha(x_2)\}$$

Appears in:

- Exact completions.
- Tripos-to-topos.
- Setoid models of extensional type theories.

From indexed posets to fibrations

In a proof-relevant logical system, derivations are labelled by proof-terms:

$$x : A \mid u : \varphi \vdash p(u) : \psi$$

Semantically, replace indexed posets with indexed categories.

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{P} \mathcal{C}at$$

From indexed posets to fibrations

In a proof-relevant logical system, derivations are labelled by proof-terms:

$$x : A \mid u : \varphi \vdash p(u) : \psi$$

Semantically, replace indexed posets with fibrations.

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{P} \mathcal{C}at \qquad \longmapsto \qquad \begin{array}{c} \int P \\ \big\downarrow \pi_P \\ \mathcal{C} \end{array}$$

$$X \longmapsto \mathcal{E}_X := K^{-1}(X) \qquad \longleftarrow\!\!\mid \qquad \begin{array}{c} \mathcal{E} \\ \big\downarrow K \\ \mathcal{C} \end{array}$$

Faithful fibrations are (equivalent to) indexed preorders.

Elementary fibrations

A fibration $K\colon \mathcal{E} \longrightarrow \mathcal{B}$ is elementary if it has finite products, *i.e.*

- $\mathcal{B}$ has finite products.
- Every fibre $\mathcal{E}_X$ has finite products.

$\Leftrightarrow$

- Both $\mathcal{B}$ and $\mathcal{E}$ have finite products.
- K preserves finite products.

and, for every Z, X in $\mathcal{B}$, there is

$$\mathcal{E}_{Z\times X} \underset{\mathrm{pr}_{1,2,2}{}^*}{\overset{\exists_{Z,X}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{E}_{Z\times X\times X}$$

where $\mathrm{pr}_{1,2,2} = \langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2\rangle\colon Z \times X \longrightarrow Z \times X \times X$, and the left adjoints satisfy:

- Frobenius Reciprocity, and
- the Beck-Chevalley Condition.

Elementary fibrations - Examples

1. The fibration obtained from an elementary doctrine is an elementary fibration.

2. The fibration $\mathrm{Fam}(\mathcal{C}) \longrightarrow \mathcal{S}et$, when $\mathcal{C}$ has finite products and a strict initial object.

3. The fibration $\mathrm{cod}_{\mathcal{C}} \colon \mathcal{C}^2 \longrightarrow \mathcal{C}$, when $\mathcal{C}$ has finite limits.

4. The fibration $\mathrm{cod}|_{\mathcal{M}} \colon \mathcal{M} \longrightarrow \mathcal{C}$, when $(\mathcal{E}, \mathcal{M})$ is a (suitable) orthogonal factorisation system on $\mathcal{C}$.

5. $\mathrm{SCIsoFib} \longrightarrow \mathcal{C}at^2 \xrightarrow{\mathrm{cod}} \mathcal{C}at$, where $\mathrm{SCIsoFib}$ = split cloven isofibrations and morphisms preserving the cleavage.
(Cf. Hofmann–Streicher's groupoid model of Martin-Löf type theory.)

Transporters

$K\colon \mathcal{E} \longrightarrow \mathcal{B}$ fibration with products has productive transporters if:

1. for every X in $\mathcal{B}$, there is a loop at X:

$$\top_X \xrightarrow{\delta_X} \mathrm{I}_X \qquad \stackrel{K}{\longmapsto} \qquad X \xrightarrow{\mathrm{pr}_{1,1}} X \times X$$

2. for every $A \in \mathcal{E}_X$, there is a carrier c_A for I_X at A:

$$\begin{array}{ccc} (\mathrm{pr_1}^* A) \wedge \mathrm{I}_X & \xrightarrow{\mathrm{c}_A} & A \\ & {\scriptstyle\delta_A}\nwarrow \quad \nearrow{\scriptstyle\mathrm{id}} & \\ & A & \end{array} \qquad \stackrel{K}{\longmapsto} \qquad \begin{array}{ccc} X \times X & \xrightarrow{\mathrm{pr}_2} & X \\ & {\scriptstyle\mathrm{pr}_{1,1}}\nwarrow \quad \nearrow{\scriptstyle\mathrm{id}} & \\ & X & \end{array}$$

where δ_A is constructed from δ_X, and

3. for every X, Y in $\mathcal{B}$, there is a vertical arrow $\chi_{X,Y}$ s.t.

$$\begin{array}{ccc} \mathrm{I}_X \boxtimes \mathrm{I}_Y & \xrightarrow{\chi_{X,Y}} & \mathrm{I}_{X \times Y} \\ & {\scriptstyle\delta_X \boxtimes \delta_Y}\nwarrow \quad \nearrow{\scriptstyle\delta_{X \times Y}} & \\ & \top_X \boxtimes \top_Y \cong \top_{X \times Y} & \end{array}$$

Transporters - Examples

1. Every elementary fibration has productive transporters.

   A faithful fibration ($\simeq$ indexed preorder) with finite products is elementary iff it has productive transporters.

2. Every model of Martin-Löf type theory has productive transporters.

   The carrier at a type $x : X \mid A(x)$ is given by the elimination rule of the identity type on X.

3. The fibration $\mathrm{cod}|_{\mathcal{R}}$ from a (suitable) weak factorisation system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ has productive transporters.

A characterisation of elementary fibrations

Theorem[3]

A fibration with products is elementary if and only if

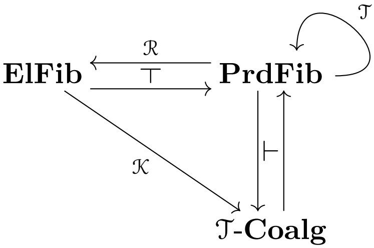
1. it has productive transporters, and
2. all arrows obtained pairing a loop $\delta_X \colon \top_X \longrightarrow \mathrm{I}_X$ with certain cartesian arrows over $\mathrm{pr}_{1,2,2} \colon Z \times X \longrightarrow Z \times X \times X$ are locally epic.

Where an arrow $\varphi \colon A \longrightarrow A'$ is locally epic if the function $(-) \circ \varphi \colon \mathcal{E}_{KB}(A', B) \longrightarrow \mathcal{E}_{K\varphi}(A, B)$ is injective for every $B \in \mathcal{E}_{KB}$.

"Proof terms that do not change the context are determined by pre-composition with φ".

[3] J.E., F. Pasquali, G. Rosolini. A characterisation of elementary fibrations. *Ann. Pure Appl. Logic* 2022.

Elementary fibrations as coalgebras



Theorem (E.–Pasquali–Rosolini)

- $\mathcal{T}(K)$ is an elementary fibration and $\mathcal{R}(K) = \mathcal{T}(K)$.
- The comonad $\mathcal{T}$ is lax-idempotent.
- The canonical comparison 2-functor $\mathcal{K}$ is a biequivalence.

The underlying 2-functor of the 2-comonad $\mathcal{T}$

For $K\colon \mathcal{E} \longrightarrow \mathcal{C}$ a fibration with finite products,
the fibration with finite products $\mathcal{T}(K)\colon \mathcal{D}es_K \longrightarrow K\text{-}\mathcal{G}pd$ is defined as follows.

$K\text{-}\mathcal{G}pd$ is the full subcategory of $\mathcal{G}pd(\mathcal{E})$ on those groupoids $\mathbf{X}$ s.t.

$$
\begin{array}{ccccc}
\mathcal{E} & \qquad & \top_X \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \bar{X} \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}\; \bar{X} \times_{\top_X} \bar{X} \\
{\scriptstyle K}\big\downarrow & & \\
\mathcal{B} & & X \;\substack{\xleftarrow{\mathrm{pr}_1} \\ \xrightarrow{\mathrm{pr}_{1,1}} \\ \xleftarrow[\mathrm{pr}_2]{}}\; X \times X \;\substack{\xleftarrow{\mathrm{pr}_{1,2}} \\ \xleftarrow{\mathrm{pr}_{1,3}} \\ \xleftarrow[\mathrm{pr}_{2,3}]{}}\; X \times X \times X
\end{array}
$$

$\mathcal{D}es_K(\mathbf{X})$ is a category of descent data for $\mathbf{X}$:
its objects are pairs (A, α) where $A \in \mathcal{E}_X$ and $\mathrm{pr}_1^* A \wedge \bar{X} \xrightarrow{\alpha} \mathrm{pr}_2^* A$
plus equations.

$\mathcal{D}es_K(\mathbf{X})$ is defined as a full subcategory of the category of algebras for a monad on K, generated by $\mathbf{X}$, in the 2-category $\mathbf{Fib}$.

To conclude:

In elementary doctrines, the equality predicate on X can be characterised as a reflexive binary relation Eq_X on X such that

$$A(x) \wedge \mathrm{Eq}_X(x, x') \vdash A(x')$$

for every predicate A over X (Leibniz's Indiscernibility of Identicals).

In elementary fibrations, Eq_X is a groupoid on X and Indiscernibility of Identicals has witnesses

$$\mathrm{d}_A\colon A(x) \wedge \mathrm{Eq}_X(x, x') \longrightarrow A(x')$$

which are algebras for a monad generated by the groupoid Eq_X.

The cofree elementary fibrations $\mathcal{T}(K)\colon \mathcal{D}es_K \longrightarrow K\text{-}\mathcal{G}pd$ are fibrations over categories of groupoids and each fibre over a groupoid consists of descent data for the groupoid.
The elementary fibrations are the subfibrations of cofree ones "closed under Indiscernibility of Identicals".