THE MONOTONE-LIGHT FACTORIZATION FOR 2-CATEGORIES VIA 2-PREORDERS

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ABSTRACT. It is shown that the reflection $2Cat \to 2Preord$ of the category of all 2-categories into the category of 2-preorders determines a monotone-light factorization system on $2Cat$ and that the light morphisms are precisely the 2-functors faithful on 2-cells with respect to the vertical structure. In order to achieve such result it was also proved that the reflection $2Cat \to 2Preord$ has stable units, a stronger condition than admissibility in categorical Galois theory, and that the 2-functors surjective both on horizontally and on vertically composable triples of 2-cells are effective descent morphisms in $2Cat$.

1. INTRODUCTION

1.1. Every map $\alpha : A \to B$ of compact Hausdorff spaces has a factorization $\alpha = me$ such that $m : C \to B$ has totally disconnected fibres and $e : A \to C$ has only connected ones. This is known as the classical monotone-light factorization of S. Eilenberg [3] and G. T. Whyburn [10].

Consider now, for an arbitrary functor $\alpha : A \to B$, the factorization $\alpha = me$ such that m is a faithful functor and e is a full functor bijective on objects. This familiar factorization for categories is as well monotone-light. Meaning that both factorizations are special and very similar cases of categorical monotone-light factorization in an abstract category $\mathbb{C}$, with respect to a full reflective subcategory $\mathbb{X}$, as was studied at [1]. What we shall show is that there is also a monotone-light factorization for 2-categories, very similar to the one given before for categories if one ignores the horizontal composition of 2-cells.

It is well known that any full reflective subcategory $\mathbb{X}$ of a category $\mathbb{C}$ gives rise, under mild conditions, to a factorization system $(\mathcal{E}, \mathcal{M})$. Hence, each of the three reflections $CompHaus \to Prof$, of compact Hausdorff spaces into Stone(profinite) spaces, $Cat \to Preord$, of categories into preorders, and now $2Cat \to 2Preord$, of 2-categories into 2-preorders yields its own reflective factorization system.

Moreover, the process of simultaneously stabilizing $\mathcal{E}$ and localizing $\mathcal{M}$, in the sense of [1], was already known to produce a new

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non reflective and stable factorization system $(\mathcal{E}', \mathcal{M}^*)$ for the adjunctions $CompHaus \to Prof$ and $Cat \to Preord$. Which is just the $(\mathbf{Monotone}, \mathbf{Light})$-factorization mentioned above. But this process does not work in general, being the monotone-light factorizations for the reflections $CompHaus \to Prof$ and $Cat \to Preord$ two rare examples. Nevertheless, we shall prove that the (**Full on 2-Cells and Bijective on Objects and Morphisms**, **Faithful on 2-Cells**)-factorization[1] for 2-categories is another instance of a successful simultaneous stabilization and localization.

What guarantees the success is the following pair of conditions, which hold in the three cases:

(1) the reflection $I : \mathbb{C} \to \mathbb{X}$ has stable units (in the sense of [2]);
(2) for each object B in $\mathbb{C}$, there is a monadic extension[2] (E, p) of B such that E is in the full subcategory $\mathbb{X}$.

Indeed, the two conditions (1) and (2) trivially imply that the $(\mathcal{E}, \mathcal{M})$-factorization is locally stable, which is a necessary and sufficient condition for $(\mathcal{E}', \mathcal{M}^*)$ to be a factorization system (see the central result of [1]).

Actually, we shall prove that the reflection $2Cat \to 2Preord$ has also stable units, as well as the reflections $Cat \to Preord$ and $CompHaus \to Prof$ were known to have. And, for the reflection $2Cat \to 2Preord$, the monadic extension (E, p) of B may be chosen to be the obvious projection from the coproduct $E = 2Cat(vh\mathbf{4}, B) \cdot vh\mathbf{4}$ of sufficiently many copies of the 2-preorder $vh\mathbf{4}$ (cf. its definition in Example 4.1), one copy for each triple of composable 2-cells in B. As for $Cat \to Preord$ and for $CompHaus \to Prof$, it was chosen to be the obvious projection from the coproduct $E = Cat(\mathbf{4}, B) \cdot \mathbf{4}$ of sufficiently many copies of the ordinal number $\mathbf{4}$, and the canonical surjection from the Stone-Čech compactification $E = \beta|B|$ of the underlying set of B, respectively. In the three cases these monadic extensions are precisely the counit morphisms of the following adjunctions from Set: the unique (up to an isomorphism) adjunction $2Cat(vh\mathbf{4}, -) \vdash (-) \cdot vh\mathbf{4} : Set \to 2Cat$ which takes the terminal object 1 to the 2-preorder $vh\mathbf{4}$; the unique (up to an isomorphism) adjunction $Cat(\mathbf{4}, -) \vdash (-) \cdot \mathbf{4} : Set \to Cat$ which takes the terminal object 1 to the ordinal number $\mathbf{4}$, and the adjunction $| \cdot | \vdash \beta : Sets \to CompHaus$, where the standard forgetful functor $| \cdot |$ is monadic, respectively.

1.2. The three reflections may be considered as admissible Galois structures[3], in the sense of categorical Galois theory, since having stable

[1] Notice that "full" and "faithful" here are with respect to the vertical composition.

[2] It is said that (E, p) is a monadic extension of B, or that p is an effective descent morphism, if the pullback functor $p^* : \mathbb{C}/B \to \mathbb{C}/E$ is monadic.

[3] In which all morphisms are considered.

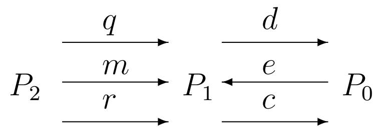
units implies admissibility. Therefore, in the three cases, for every object B in $\mathbb{C}$, we know that the full subcategory $TrivCov(B)$ of $\mathbb{C}/B$, determined by the trivial coverings of B (i.e., the morphisms over B in $\mathcal{M}$), is equivalent to $\mathbb{X}/I(B)$. Moreover, the categorical form of the fundamental theorem of Galois theory gives us even more information on each $\mathbb{C}/B$ using the subcategory $\mathbb{X}$. It states that the full subcategory $Spl(E,p)$ of $\mathbb{C}/B$, determined by the morphisms split by the monadic extension (E,p) of B, is equivalent to the category $\mathbb{X}^{Gal(E,p)}$ of internal actions of the Galois pregroupoid of (E,p). In fact, the conditions (1) and (2) above imply that $Gal(E,p)$ is really an internal groupoid in $\mathbb{X}$ (see section 5.3 of [1]).

The condition (1) implies as well that any covering over an object which belongs to the subcategory is just a trivial covering. An easy consequence of this last statement, condition (2) and the fact that coverings are pullback stable, is that a covering morphism $\alpha : A \to B$ is such if and only if, for every morphism $\phi : X \to B$ with X in the subcategory $\mathbb{X}$, the pullback $X \times_B A$ of α along ϕ is also in $\mathbb{X}$. In particular, when the reflection has stable units, a monadic extension (E,p), as in condition (2), is a covering if and only if the kernel pair of p is in the full subcategory $\mathbb{X}$ of $\mathbb{C}$. Thus, since the monadic extensions considered for the three cases are in fact coverings, we conclude that $Gal(2Cat(vh\mathbf{4},B)\cdot vh\mathbf{4},p)$, $Gal(Cat(\mathbf{4},B)\cdot \mathbf{4},p)$ and $Gal(\beta|B|,p)$ are not just internal groupoids, but internal equivalence relations in $2Preord$, $Preord$ and $CompHaus$, respectively.

2. The category of all 2-categories

Consider the category $2Cat$, with objects all 2-categories and whose morphisms are the 2-functors (see [6, §XII.3]). Its definition is going to be stated in a way that suits our purposes. In order to do so, some intermediate structures need to be defined previously.

First, consider the category $\mathbb{P}$ generated by the following *precategory diagram*,



in which

$d\circ e = 1_{P_0} = c\circ e,\ d\circ m = d\circ q,\ c\circ m = c\circ r$ *and* $c\circ q = d\circ r,$
where 1_{P_0} stands for the identity morphism of P_0 (see [1, §4.1]).

A *precategory* is an object in the category of presheaves $\hat{\mathbb{P}} = Set^{\mathbb{P}}$, that is, any functor $P : \mathbb{P} \to Set$ to the category of sets.

If
$$Q_2 \;\begin{array}{c} \xrightarrow{\;q'\;} \\ \xrightarrow{\;m'\;} \\ \xrightarrow{\;r'\;} \end{array}\; Q_1 \;\begin{array}{c} \xrightarrow{\;d'\;} \\ \xleftarrow{\;e'\;} \\ \xrightarrow{\;c'\;} \end{array}\; Q_0$$

is another precategory diagram, then a triple (f_2, f_1, f_0) with $f_2 : P_2 \to Q_2$, $f_1 : P_1 \to Q_1$ and $f_0 : P_0 \to Q_0$, will be called a *precategory morphism diagram* provided the following equations hold: $f_0 \circ d = d' \circ f_1$, $f_0 \circ c = c' \circ f_1$, $f_1 \circ e = e' \circ f_0$, $f_1 \circ q = q' \circ f_2, f_1 \circ m = m' \circ f_2$, $f_1 \circ r = r' \circ f_2$.

Secondly, consider the category $2\mathbb{P}$ generated by the following *2-precategory diagram*,

$$\begin{array}{ccccc}
hvP_2 & \begin{array}{c} \xrightarrow{\;hq\times hq\;} \\ \xrightarrow{\;hm\times hm\;} \\ \xrightarrow{\;hr\times hr\;} \end{array} & vP_2 & \begin{array}{c} \xrightarrow{\;hd\times hd\;} \\ \xleftarrow{\;he\times he\;} \\ \xrightarrow{\;hc\times hc\;} \end{array} & P_0 \\
vr^2 \downarrow \; vm^2 \downarrow \; vq^2 \downarrow & & vr \downarrow \; vm \downarrow \; vq \downarrow & & \downarrow 1_{P_0} \\
hP_2 & \begin{array}{c} \xrightarrow{\;hq\;} \\ \xrightarrow{\;hm\;} \\ \xrightarrow{\;hr\;} \end{array} & 2P_1 & \begin{array}{c} \xrightarrow{\;hd\;} \\ \xleftarrow{\;he\;} \\ \xrightarrow{\;hc\;} \end{array} & P_0 \\
vc^2 \downarrow \; ve^2 \uparrow \; vd^2 \downarrow & & vc \downarrow \; ve \uparrow \; vd \downarrow & & \downarrow 1_{P_0} \\
P_2 & \begin{array}{c} \xrightarrow{\;q\;} \\ \xrightarrow{\;m\;} \\ \xrightarrow{\;r\;} \end{array} & P_1 & \begin{array}{c} \xrightarrow{\;d\;} \\ \xleftarrow{\;e\;} \\ \xrightarrow{\;c\;} \end{array} & P_0 \;,
\end{array} \qquad (2.1)$$

in which:

- each one of the three horizontal diagrams (upwards, P, hP and hvP) is a precategory diagram;
- each one of the three vertical diagrams (from the left to the right, vhP, vP and the trivial P_0) is a precategory diagram;
- $(vc^2, vc, 1_{P_0})$, $(ve^2, ve, 1_{P_0})$, $(vd^2, vd, 1_{P_0})$, $(vr^2, vr, 1_{P_0})$, $(vm^2, vm, 1_{P_0})$, $(vq^2, vq, 1_{P_0})$ are all six precategory morphism diagrams (equivalently, $(hq \times hq, hq, q)$, $(hm \times hm, hm, m)$, $(hr \times hr, hr, r)$, $(hd \times hd, hd, d)$, $(he \times he, he, e)$, $(hc \times hc, hc, c)$ are all six precategory morphism diagrams).

Notice that the names given to objects and morphisms in (2.1) are arbitrary, being so chosen in order to relate to the following last definition of section 2 (for instance, $vq^2 = vq \times vq$ will denote the morphism uniquely determined by a pullback diagram).

The category $2Cat$ of all 2-categories is the full subcategory of $\hat{2\mathbb{P}} = Set^{2\mathbb{P}}$, determined by its objects $C : 2\mathbb{P} \to Set$ such that the image by C of each horizontal and vertical precategory diagram in (2.1) is a category. That is, for instance, in the case of the bottom horizontal precategory diagram in (2.1):

the commutative square

$$\begin{array}{ccc} C(P_2) & \xrightarrow{Cq} & C(P_1) \\ {\scriptstyle Cr}\downarrow & & \downarrow{\scriptstyle Cc} \\ C(P_1) & \xrightarrow{Cd} & C(P_0) \end{array} \qquad (2.2)$$

is a pullback diagram in Set;

the associative and unit laws hold for the operation Cm, that is, the following respective diagrams commute in Set,

$$\begin{array}{ccc} C(P_2)\times_{C(P_1)} C(P_2) & \xrightarrow{Cm\times Cq} & C(P_2) \\ {\scriptstyle Cr\times Cm}\downarrow & & \downarrow{\scriptstyle Cm} \\ C(P_2) & \xrightarrow{Cm} & C(P_1), \end{array} \qquad (2.3)$$

$$\begin{array}{ccccc} C(P_0)\times_{C(P_0)} C(P_1) & \xrightarrow{Ce\times 1_{C(P_1)}} & C(P_2) & \xleftarrow{1_{C(P_1)}\times Ce} & C(P_1)\times_{C(P_0)} C(P_0) \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle Cm} & & \downarrow{\scriptstyle pr_1} \\ C(P_1) & \xrightarrow{1_{C(P_1)}} & C(P_1) & \xleftarrow{1_{C(P_1)}} & C(P_1) \end{array} . \qquad (2.4)$$

It would be a long and trivial calculation to check that there is an isomorphism between the category of all 2-categories (in the sense of [6, §XII.3]) and the full subcategory of $\hat{2\mathbb{P}}$ just defined. Notice that: the requirement that the horizontal composite of two vertical identities is itself a vertical identity is encoded in diagram (2.1) in the commutativity of the square $hm \circ ve^2 = ve \circ m$; the interchange law, which relates the vertical and the horizontal composites of 2-cells, is encoded in diagram (2.1) in the commutativity of the square $vm \circ hm \times hm = hm \circ vm^2$.

3. Internal categories and limits

In section 2, if the category Set of sets is replaced by any category $\mathcal{C}$ with pullbacks, then one obtains the definition of $2Cat(\mathcal{C})$, the category of internal 2-categories in $\mathcal{C}$.

In this section 3, the goal is to show that the category of all 2-categories $2Cat$ is closed under limits in the presheaves category $\hat{2\mathbb{P}} = Set^{2\mathbb{P}}$. The following Lemmas 3.1 and 3.2 give some well known facts about limits of internal categories, which will translate into internal 2-categories, and finally into 2-categories in the special case of $\mathcal{C} = Set$.

In what follows, $Cat(\mathcal{C})$ will denote the category of internal categories in $\mathcal{C}$, that is, the full subcategory of the category of functors $\mathcal{C}^{\mathbb{P}}$, determined by all the functors $C : \mathbb{P} \to \mathcal{C}$ such that the diagram (2.2) is a pullback diagram in $\mathcal{C}$ and the diagrams (2.3) and (2.4) commute in $\mathcal{C}$ ($\mathbb{P}$ is of course the category defined in section 2).

Lemma 3.1. *Let $\mathcal{C}$ be a category with pullbacks.*

Then, $Cat(\mathcal{C})$ is closed under pullbacks in $\mathcal{C}^{\mathbb{P}}$, where pullbacks exist and are calculated pointwise.

Lemma 3.2. *Let $\mathcal{C}$ be a category with pullbacks.*

If $\mathbb{I}$ is a discrete category (that is, a set) and $\mathcal{C}$ has all limits $\mathbb{I} \to \mathcal{C}$, then $Cat(\mathcal{C})$ is closed under all limits $\mathbb{I} \to Cat(\mathcal{C})$ in $\mathcal{C}^{\mathbb{P}}$, where limits $\mathbb{I} \to \mathcal{C}^{\mathbb{P}}$ exist and are calculated pointwise.

Corollary 3.1. *If $\mathcal{C}$ has all limits then $2Cat(\mathcal{C})$ is closed under limits in the functor category $\mathcal{C}^{2\mathbb{P}}$, where all limits exist and are calculated pointwise.*

In particular, for $\mathcal{C} = Set$, $2Cat$ is closed under limits in $\hat{2\mathbb{P}} = Set^{2\mathbb{P}}$.

Proof. The proof follows from the fact that limits are calculated pointwise in $\mathcal{C}^{2\mathbb{P}}$, and that a category with pullbacks and all products has all limits, and fom Lemmas 3.1 and 3.2. □

4. Effective descent morphisms in 2Cat

Consider again the category of all categories Cat and its full inclusion in the category of precategories $\hat{\mathbb{P}} = Set^{\mathbb{P}}$.

A functor $p : \mathbb{E} \to \mathbb{B}$ is an effective descent morphism (e.d.m.)[4] in Cat if and only if it is surjective on composable triples of morphisms. The proof of this statement can be found in [5, Proposition 6.2]. In a completely analogous way, a class of effective descent morphisms in $2Cat$ is going to be given in the following Proposition 4.1.

Proposition 4.1. *A 2-functor $2p : 2\mathbb{E} \to 2\mathbb{B}$ is an e.d.m. in the category of all 2-categories $2Cat$ if it is surjective both on horizontally composable triples of 2-cells and on vertically composable triples of 2-cells.*

Proof. Let $2p : 2\mathbb{E} \to 2\mathbb{B}$ be surjective on triples of composable 2-cells (both horizontally and vertically). Then, $2p$ is an e.d.m. in $\hat{2\mathbb{P}} = Set^{2P}$, since the effective descent morphisms in a category of presheaves are simply those surjective pointwise (which, of course, is implied by either surjectivity on triples of composable 2-cells). Hence, the following instance of [5, Corollary 3.9] can be applied:

[4]Also called a *monadic extension* in categorical Galois theory.

if $2p : 2\mathbb{E} \to 2\mathbb{B}$ in $2Cat$ is an e.d.m. in $2\hat{\mathbb{P}} = Set^{2\mathbb{P}}$ then $2p$ is an e.d.m. in $2Cat$ if and only if , for every pullback square

$$\begin{array}{ccc} 2\mathbb{D} & \longrightarrow & 2A \\ \downarrow & & \downarrow \\ 2\mathbb{E} & \xrightarrow{\ 2p\ } & 2\mathbb{B} \end{array} \qquad (4.1)$$

in $2\hat{\mathbb{P}} = Set^{2\mathbb{P}}$ such that $2\mathbb{D}$ is in $2Cat$, then also $2A$ is in $2Cat$.

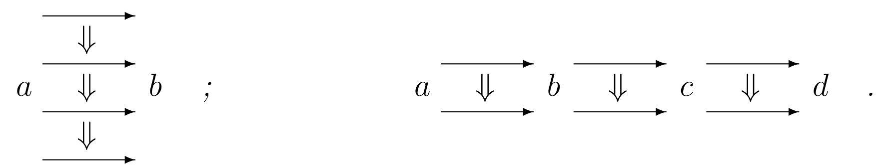
Since the pullback square (4.1) is calculated pointwise (cf. Corollary 3.1), it induces six other pullback squares in $\hat{\mathbb{P}} = Set^{\mathbb{P}}$, corresponding to the three rows P, hP and hvP, and the three columns vhP, vP and P_0, in the 2-precategory diagram (2.1).

The fact that $2p$ is surjective on triples of composable 2-cells (both horizontally and vertically) implies that its six restrictions (to the six rows and columns $2A(P)$, $2A(hP)$, $2A(hvP)$, $2A(vhP)$, $2A(vP)$ and $2A(P_0)$) are surjective on triples of composable morphisms in Cat, as it is easy to check. Hence, these six restrictions are effective descent morphisms in Cat. Therefore, $2A$ must always be a 2-category, provided so is $2\mathbb{D}$.

□

Example 4.1. *It is obvious that the coproduct $\coprod$ of 2-categories is just the disjoint union, as for categories.*

Let $v\mathbf{4}$ and $h\mathbf{4}$ be the 2-categories generated by the following two diagrams, respectively:



Consider, for each 2-category $2\mathbb{B}$, the 2-category

$$2\mathbb{E} = (\coprod_{i\in I} v\mathbf{4}) \coprod (\coprod_{j\in J} h\mathbf{4}),$$

such that I is the set of all vertically composable triples of 2-cells in $2\mathbb{B}$, and J is the set of all horizontally composable triples of 2-cells in $2\mathbb{B}$.

Then, there is an e.d.m. $2p : 2\mathbb{E} \to 2\mathbb{B}$ which projects the corresponding copy of $v\mathbf{4}$ and $h\mathbf{4}$ to every $i \in I$ and every $j \in J$, respectively.

As another option, let

$$2\mathbb{E} = \coprod_{k \in I \cup J} vh\mathbf{4},$$

with $vh\mathbf{4}$ *the 2-category*[5] *generated by the following diagram,*

$$\begin{array}{ccccccc} & \longrightarrow & & \longrightarrow & & \longrightarrow & \\ & \Downarrow & & \Downarrow & & \Downarrow & \\ & \longrightarrow & & \longrightarrow & & \longrightarrow & \\ a & \Downarrow & b & \Downarrow & c & \Downarrow & d \quad . \\ & \longrightarrow & & \longrightarrow & & \longrightarrow & \\ & \Downarrow & & \Downarrow & & \Downarrow & \\ & \longrightarrow & & \longrightarrow & & \longrightarrow & \end{array}$$

5. The reflection of 2-categories into 2-preorders has stable units and a monotone-light factorization

Let $2Preord$ be the full subcategory of $2Cat$ determined by the objects $C : 2\mathbb{P} \to Set$ such that Cvd and Cvc are jointly monic (cf. diagram (2.1)), that is,

$$C(vP_2) \underset{Cvr}{\overset{Cvq}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xrightarrow{Cvm}}}}} C(2P_1) \underset{Cvc}{\overset{Cvd}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{Cve}}}}} C(P_1) \tag{5.1}$$

is a preordered set.

There is a reflection

$$H \vdash I : 2Cat \longrightarrow 2Preord, \quad a \underset{g}{\overset{f}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow\theta}}}} b \ \mapsto \ a \underset{g}{\overset{f}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow\leq}}}} b, \tag{5.2}$$

which identifies all 2-cells which have the same domain and codomain for the vertical composition. That is, the reflector I takes the middle vertical category $C(vP)$ (cf. diagram (5.1)) to its image by the well known reflection $Cat \to Preord$ from categories into preordered sets (see [7]).

Many of the results in [9] are going to be stated again, with small improvements in their presentation[6], in order to prove that the reflection $H \vdash I : 2Cat \to 2Preord$ has stable units (in the sense of [2]).

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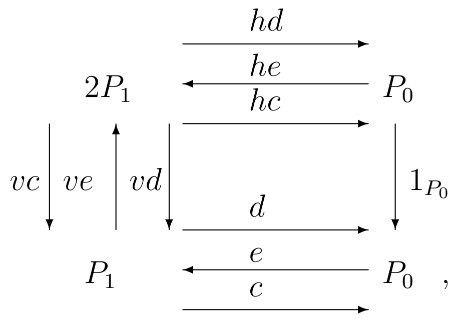
[5] Remark that $v\mathbf{4}$, $h\mathbf{4}$ and $vh\mathbf{4}$ are really 2-preorders as defined just below at the beginning of the following section 5.

[6] The reader could easily bring these small improvements to the paper [9]. In fact, although they are stated here in the particular case of the reflection from $2Cat$ into $2Preord$, they are completely general.

5.1. Ground structure. Consider the adjunction $H \vdash I : 2Cat \to 2Preord$, described just above in (5.2), with unit $\eta : 1_{2Cat} \to HI$.

- $2Cat$ has pullbacks (in fact, it has all limits - see Corollary 3.1).
- H is a full inclusion of $2Preord$ in $2Cat$, that is, I is a reflection of a category with pullbacks into a full subcategory.
- Consider also the forgetful functor $U : 2Cat \to 2RGrph$, where $2RGrph$ is the presheaves category $Set^{2\mathbb{G}}$, with $2\mathbb{G}$ the category generated by the following *2-reflexive graph diagram*,



  satisfying the same equations as in the 2-precategory diagram (2.1).
- $\mathcal{E}$ denotes the class of all morphisms $(2g_1, g_1, g_0) : G \to H$ of $2RGrph$ which are bijections on objects and on arrows, and surjections on 2-cells (that is, $g_0 : G(P_0) \to H(P_0)$ and $g_1 : G(P_1) \to H(P_1)$ are bijections, and $2g_1 : G(2P_1) \to H(2P_1)$ is a surjection).
- $\mathcal{T} = \{T\}$ is a singular set, with T the 2-preorder generated by the diagram $a \underset{h'}{\overset{h}{\rightrightarrows}} a'$ with $\Downarrow \leq$ (5.3),

  that is, a 2-preorder with two objects, two non-identity arrows and only one non-identity (both horizontally and vertically) 2-cell.

Then, the following four conditions are satisfied.

(a) U preserves pullbacks (in fact, it preserves all limits).

(b) $\mathcal{E}$ is pullback stable in $2RGrph$, and if $g' \circ g$ is in $\mathcal{E}$ so is g', provided g is in $\mathcal{E}$.[7]

[7] In [9], it was also demanded in (b) that $\mathcal{E}$ is closed under composition, which is not needed. We take this opportunity to correct that redundancy in [9].

(c) Every map $U\eta_C : U(C) \to UHI(C)$ belongs to $\mathcal{E}$, $C \in 2Cat$ (this is also obvious).

(d) [8]Let $g : N \to M$ be any morphism of $2Preord$ such that $UHg : UH(N) \to UH(M)$ is in $\mathcal{E}$.
If,
there is one morphism $f : A \to UH(N)$ of $2RGrph$ in $\mathcal{E}$ such that,
for all morphisms $c : T \to M$ in $2Preord$,
there is a commutative diagram as below

$$\begin{array}{ccccc} A \times_{UH(M)} UH(T) & \xrightarrow{pr_2} & & & UH(T) \\ {\scriptstyle pr_1}\downarrow & & & \swarrow & \downarrow{\scriptstyle UHc} \\ A & \xrightarrow[f]{} & \mathrm{UH(N)} & \xrightarrow[UHg]{} & UH(M) \end{array} \tag{5.4}$$

then
$g : N \to M$ is an isomorphism in $2Preord$.

It remains to show that the statement in (d) is true, which is trivial, since if $g : N \to M$ is in $\mathcal{E}$, seen as a morphism of $2RGrph$, then g must be an isomorphism in $2Preord$ by the uniqueness of the 2-cells in N and in M.

5.2. Stable units. Using the fact that a *ground structure* holds, it will be possible to show that $H \vdash I : 2Cat \to 2Preord$ is an admissible reflection in the sense of Galois categorical theory (cf. [4]) or, equivalently, semi-left-exact in the sense of [2]. Furthermore, it will be shown, always using the results in [9], that the reflection $H \vdash I : 2Cat \to 2Preord$ satisfies the stronger condition of having stable units.

Definition 5.1. Consider any morphism $\mu : T \to HI(C)$ from T ($\in \mathcal{T}$; cf. (5.3)), for some $C \in 2Cat$.

The *connected component* of the morphism μ is the pullback $C_\mu = C \times_{HI(C)} T$ in the following pullback square

$$\begin{array}{ccc} C_\mu & \xrightarrow{\pi_2^\mu} & T \\ {\scriptstyle \pi_1^\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ C & \xrightarrow{\eta_C} & HI(C) \end{array} \quad (5.5) \quad ,$$

where η_C is the unit morphism of C in the reflection $H \vdash I : 2Cat \to 2Preord$, and T is identified with $H(T)$.

[8]This item is rephrased from [9], in a way that seems to us now more easily understandable. Remark also that the diagram (5.4) is simplified, suppressing one morphism $UH(T) \to UH(T)$, which can be the identity. We take this opportunity to correct that other redundancy.

Theorem 5.1. *The reflection $H \vdash I : 2Cat \to 2Preord$ is semi-left-exact.*

Proof. According to Theorem 2.1 in [9], one has to show that $I\pi_2^\mu : I(C_\mu) \to I(T)$ is an isomorphism, for every connected component C_μ.

If $\mu\Big(a \overset{h}{\underset{h'}{\rightrightarrows}}\!\Downarrow\leq\; a'\Big) = c \overset{k}{\underset{k'}{\rightrightarrows}}\!\Downarrow\leq\; c'$, then,

since $U\eta_C \in \mathcal{E}$ (identity on objects and morphisms, and surjection on 2-cells), the pullback C_μ is the 2-category generated by the diagram

$$(c,a) \overset{(k,h)}{\underset{(k',h')}{\rightrightarrows}} \Downarrow(\theta_r, \leq)\; (c',a') \quad ,$$

with $\theta_r \in Hom_{C(vP)}(k,k') = \{\theta_r \mid r \in R\}$, that is, with θ_r any 2-cell with domain k and codomain k'.

Hence, $I(C_\mu) \cong T$. □

Theorem 5.2. *The reflection $H \vdash I : 2Cat \to 2Preord$ has stable units.*

Proof. According to Theorem 2.2 in [9], one has to show that $I(C_\mu \times_T D_\nu) \cong T$, for every pair of connected components C_μ, D_ν, where $C_\mu \times_T D_\nu$ is the pullback object in any pullback of the form

$$\begin{array}{ccc} C_\mu \times_T D_\nu & \xrightarrow{p_2} & D_\nu \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_2^\nu} \\ C_\mu & \xrightarrow{\pi_2^\mu} & T \end{array} \quad ,$$

where π_2^μ and π_2^ν are the second projections in pullback diagrams of the form (5.5).

According to the previous Theorem 5.1, one can suppose (up to isomorphism) that $C_\mu = c \overset{k}{\underset{k'}{\rightrightarrows}}\!\Downarrow\theta_r\; c'$, $r \in R$, and $D_\nu = d \overset{l}{\underset{l'}{\rightrightarrows}}\!\Downarrow\delta_s\; d'$, $s \in S$ (the identity morphisms and the identity 2-cells are not displayed). Hence, $C_\mu \times_T D_\nu = (c,d) \overset{(k,l)}{\underset{(k',l')}{\rightrightarrows}} \Downarrow(\theta_r, \delta_s)\; (c',d')$, $(r,s) \in$

$R \times S$, and so it is obvious that $I(C_\mu \times_T D_\nu) \cong a \overset{h}{\underset{h'}{\overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow\leq}}}} a'$. □

5.3. Monotone-light factorization for 2-categories via 2-preorders.

Theorem 5.3. *The reflection $H \vdash I : 2Cat \to 2Preord$ does have a monotone-light factorization.*

Proof. The statement is a consequence of the central result of [1] (cf. Corollary 6.2 in [8]), because $H \vdash I$ has stable units (cf. Theorem 5.2) and for every $2\mathbb{B} \in 2Cat$ there is an e.d.m. $2p : 2\mathbb{E} \to 2\mathbb{B}$ with $2\mathbb{E} \in 2Preord$ (cf. Example 4.1).

□

In the following section 6, it will be proved that the monotone-light factorization system is not trivial. That is, it does not coincide with the reflective factorization system associated to the reflection of $2Cat$ into $2Preord$.

6. Vertical and stably-vertical 2-functors

In this section, it will be given a characterization of the class of vertical morphisms $\mathcal{E}_I$ in the reflective factorization system $(\mathcal{E}_I, \mathcal{M}_I)$, and of the class of the stably-vertical morphisms $\mathcal{E}'_I$ $(\subseteq \mathcal{E}_I)$[9] in the monotone-light factorization system $(\mathcal{E}'_I, \mathcal{M}^*_I)$, both associated to the reflection $2Cat \to 2Preord$. Then, since $\mathcal{E}'_I$ is a proper class of $\mathcal{E}_I$, one concludes that $(\mathcal{E}'_I, \mathcal{M}^*_I)$ is a non-trivial monotone-light factorization system.

Consider a 2-functor $f : A \to B$, which is obviously determined by the three functions $f_0 : A(P_0) \to B(P_0)$, $f_1 : A(P_1) \to B(P_1)$ and $2f_1 : A(2P_1) \to B(2P_1)$ (cf. diagram (2.1)), so that we may make the identification $f = (2f_1, f_1, f_0)$.

Proposition 6.1. *A 2-functor $f = (2f_1, f_1, f_0) : A \to B$ belongs to the class $\mathcal{E}_I$ of vertical 2-functors if and only if the following two conditions hold:*

(1) *f_0 and f_1 are bijections;*
(2) *for every two elements h and h' in $A(P_1)$, if $Hom_{B(vP)}(f_1h, f_1h')$ is nonempty then so is $Hom_{A(vP)}(h, h')$.*

Proof. The 2-functor $f = (2f_1, f_1, f_0)$ belongs to $\mathcal{E}_I$ if and only if If is an isomorphism (cf. [1, §3.1]), that is, If_0, If_1, and $I2f_1$ are bijections. Since $If_0 = f_0$ and $If_1 = f_1$, the fact that $f \in \mathcal{E}_I$ implies and is implied by (1) and (2) is trivial.

□

[9] $\mathcal{E}'_I$ is the largest subclass of $\mathcal{E}_I$ stable under pullbacks.

Proposition 6.2. *A 2-functor $f = (2f_1, f_1, f_0) : A \to B$ belongs to the class $\mathcal{E}'_I$ of stably-vertical 2-functors if and only if the following two conditions hold:*

1. *f_0 and f_1 are bijections;*
2. *for every two elements h and h' in $A(P_1)$, f induces a surjection $Hom_{A(vP)}(h, h') \to Hom_{B(vP)}(f_1h, f_1h'))$ (f is a "full functor on 2-cells").*

Proof. As every pullback $g^*(f) = \pi_1 : C \times_B A \to C$ in $2Cat$ of f along any 2-functor $g : C \to B$ is calculated pointwise, and $(2f_1, f_1) : A(vP) \to B(vP)$ is a stably-vertical functor for the reflection $Cat \to Preord$, that is, f_1 is a bijection and $(2f_1, f_1)$ is a full functor (cf. Propositions 4.4 and 3.2 in [7]), then (1) and (2) imply that $g^*(f)$ belongs to $\mathcal{E}_I$ (cf. last Proposition 6.1).

Hence, $f \in \mathcal{E}'_I$ if (1) and (2) hold.

If $f \in \mathcal{E}'_I$, then $f \in \mathcal{E}_I$ ($\mathcal{E}'_I \subseteq \mathcal{E}_I$), and therefore (1) holds.

Suppose now that (2) does not hold, so that there is $\theta : f_1h \to f_1h'$ not in the image of f, and consider the 2-category C generated by the diagram $b \;\substack{\xrightarrow{f_1h} \\ \Downarrow\theta \\ \xrightarrow[f_1h']{}}\; b'$, and let g be the inclusion of C in B. Then,

$C \times_B A \cong b \;\substack{\xrightarrow{f_1(h)} \\ \\ \xrightarrow[f_1(h')]{}}\; b'$, with no non-identity 2-cells, and so $g^*(f)$ is not in $\mathcal{E}_I$.

Hence, if $f \in \mathcal{E}'_I$ then (1) and (2) must hold. □

It is evident that $\mathcal{E}'_I$ is a proper class of $\mathcal{E}_I$, therefore the monotone-light factorization system $(\mathcal{E}'_I, \mathcal{M}^*_I)$ is non-trivial ($\neq (\mathcal{E}_I, \mathcal{M}_I)$).

7. Trivial coverings for 2-categories via 2-preorders

A 2-functor $f : A \to B$ belongs to the class $\mathcal{M}_I$ of trivial coverings (with respect to the reflection $H \vdash I : 2Cat \to 2Preord$) if and only if the following square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & I(A) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle If} \\ B & \xrightarrow{\eta_B} & I(B) \end{array} \qquad (7.1)$$

is a pullback diagram, where η_A and η_B are unit morphisms for the reflection $H \vdash I : 2Cat \to 2Preord$ (cf. [2, Theorem 4.1]).

Since the pullback (as any limit) is calculated pointwise in $2Cat$ (cf. Corollary 3.1), then $f \in \mathcal{M}_I$ if and only if the following seven squares are pullbacks, corresponding to the seven pointwise components of η_A and of η_B (cf. diagram (2.1)):

$$\begin{array}{ccc} A(P_i) & \xrightarrow{\eta_{A(P_i)}} & I(A)(P_i) \\ {\scriptstyle f_{P_i}}\downarrow & (D_i) & \downarrow{\scriptstyle If_{P_i}} \\ B(P_i) & \xrightarrow{\eta_{B(P_i)}} & I(B)(P_i) \end{array} \qquad (i = 0, 1, 2)$$

these three squares are pullbacks since $\eta_{A(P_i)}$ and $\eta_{B(P_i)}$ are identity maps for $i = 0, 1, 2$ (cf. diagram (2.1) and the definition of the reflection $H \vdash I : 2Cat \to 2Preord$ in (5.2));

$$\begin{array}{ccc} A(2P_1) & \xrightarrow{\eta_{A(2P_1)}} & I(A)(2P_1) \\ {\scriptstyle f_{2P_1}}\downarrow & (2D) & \downarrow{\scriptstyle If_{2P_1}}; \\ B(2P_1) & \xrightarrow{\eta_{B(2P_1)}} & I(B)(2P_1) \end{array} \qquad \begin{array}{ccc} A(vP_2) & \xrightarrow{\eta_{A(vP_2)}} & I(A)(vP_2) \\ {\scriptstyle f_{vP_2}}\downarrow & (vD) & \downarrow{\scriptstyle If_{vP_2}}; \\ B(vP_2) & \xrightarrow{\eta_{B(vP_2)}} & I(B)(vP_2) \end{array}$$

$$\begin{array}{ccc} A(hP_2) & \xrightarrow{\eta_{A(hP_2)}} & I(A)(hP_2) \\ {\scriptstyle f_{hP_2}}\downarrow & (hD) & \downarrow{\scriptstyle If_{hP_2}}; \\ B(hP_2) & \xrightarrow{\eta_{B(hP_2)}} & I(B)(hP_2) \end{array} \qquad \begin{array}{ccc} A(hvP_2) & \xrightarrow{\eta_{A(hvP_2)}} & I(A)(hvP_2) \\ {\scriptstyle f_{hvP_2}}\downarrow & (hvD) & \downarrow{\scriptstyle If_{hvP_2}} \\ B(hvP_2) & \xrightarrow{\eta_{B(hvP_2)}} & I(B)(hvP_2). \end{array}$$

Notice that if diagram (2.1) is restricted to the (vertical) precategory diagram vP, one obtains from (7.1) the following square in Cat, with unit morphisms of the reflection of all categories into preorders $Cat \to Preord$ (cf. [7]),

$$\begin{array}{ccc} A(vP) & \xrightarrow{\eta_{A(vP)}} & I(A(vP)) \\ {\scriptstyle f_{vP}}\downarrow & & \downarrow{\scriptstyle If_{vP}} \\ B(vP) & \xrightarrow{\eta_{B(vP)}} & I(B(vP)). \end{array} \qquad (7.2)$$

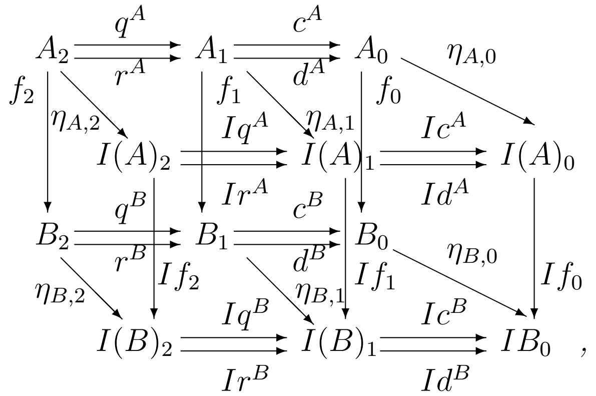
It is known (cf. [7, Proposition 3.1]) that this square is a pullback in Cat if and only if, for every two objects h and h' in $A(P_1)$ with $Hom_{A(2P_1)}(h, h')$ nonempty, the map

$$Hom_{A(2P_1)}(h, h') \to Hom_{B(2P_1)}(f_1 h, f_1 h')$$

induced by f is a bijection.

A necessary condition for the 2-functor f to be a trivial covering was just stated; the following Lemma 7.1 will help to show that this necessary condition is also sufficient in next Proposition 7.1.

Lemma 7.1. *Consider the following commutative parallelepiped*



where the five squares $d^A q^A = c^A r^A$, $d^B q^B = c^B r^B$, $Id^A Iq^A = Ic^A Ir^A$, $If_0 \eta_{A,0} = \eta_{B,0} f_0$ *and* $If_1 \eta_{A,1} = \eta_{B,1} f_1$ *are pullbacks.*

Then, the square $If_2 \eta_{A,2} = \eta_{B.2} f_2$ *is also a pullback.*[10]

Proof. The proof is obtained by an obvious diagram chase. □

Proposition 7.1. *A 2-functor* $f : A \to B$ *is a trivial covering for the reflection* $H \vdash I : 2Cat \to 2Preord$ *(in notation,* $f \in \mathcal{M}_I$*) if and only if, for every two objects* h *and* h' *in* $A(P_1)$ *with* $Hom_{A(2P_1)}(h, h')$ *nonempty, the map*

$$Hom_{A(2P_1)}(h, h') \to Hom_{B(2P_1)}(f_1 h, f_1 h')$$

induced by f *is a bijection.*

Proof. In the considerations just above, it was showed that the statement warrants that the squares $(2D)$ and (vD) are pullbacks, adding to the fact that (D_0), (D_1) and (D_2) are all the three pullbacks.

Then, (hD) and (hvD) must also be pullbacks according to Lemma 7.1. □

8. Coverings for 2-categories via 2-preorders

A 2-functor $f : A \to B$ belongs to the class $\mathcal{M}_I^*$ of coverings (with respect to the reflection $H \vdash I : 2Cat \to 2Preord$) if there is some effective descent morphism (also called monadic extension in Galois categorical theory) $p : C \to B$ in $2Cat$ with codomain B such that the pullback $p^*(f) : C \times_B A \to C$ of f along p is a trivial covering ($p^*(f) \in \mathcal{M}_I$).

[10] The notation used in diagram (7.3) is arbitrary, being so chosen in order to make the application of Lemma 7.1 in this section more easily understandable.

The following Lemma 8.1 can be found in [7, Lemma 4.2], in the context of the reflection of categories into preorders, but for 2-categories via 2-preorders the proof is exactly the same, since the same conditions hold (cf. Theorem 5.2 and Example 4.1). The next Proposition 8.1 characterizes the coverings for 2-categories via 2-preorders.

Lemma 8.1. *A 2-functor $f : A \to B$ in $2Cat$ is a covering (for the reflection $H \vdash I : 2Cat \to 2Preord$) if and only if, for every 2-functor $\varphi : X \to B$ over B from any 2-preorder X, the pullback $X \times_B A$ of f along φ is also a 2-preorder.*

Proposition 8.1. *A 2-functor $f : A \to B$ in $2Cat$ is a covering (for the reflection $H \vdash I : 2Cat \to 2Preord$) if and only if it is faithful vertically with respect to 2-cells, that is, for every pair of morphisms g and g', the map*

$$Hom_{A(2P_1)}(g, g') \to Hom_{B(2P_1)}(f_1 g, f_1 g'$$

induced by f is an injection.

Proof. Consider again the 2-preorder T generated by the diagram $a \overset{h}{\underset{h'}{\rightrightarrows}} a'$ with $\Downarrow \leq$.

If f is not faithful vertically with respect to 2-cells, then, by including T in B, one could obtain a pullback $T \times_B A$ that is not a preorder.

Therefore, f is not a covering, by the previous Lemmma 8.1.

Reciprocally, consider any 2-functor $\varphi : X \to B$ such that X is a 2-preorder.

If f is faithful (vertically with respect to 2-cells), then the pullback $X \times_B A$ is a 2-preorder, given the nature of X. Hence, f is a covering, by the previous Lemmma 8.1.

□

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