The small index property of the Fraïssé limit of finite Heyting algebras

Kentarô Yamamoto

Institute of Computer Science, the Czech Academy of Sciences

June 19, 2022

Motivation: Problem of reconstruction

Problem

For a reasonable class $\mathcal{K}$ of (countable) structures and $M, N \in \mathcal{K}$:

$$\mathrm{Aut}(M) \cong \mathrm{Aut}(N) \overset{?}{\iff} M \cong N$$

This is a way to precisify the broader question:

Problem

Is it possible to describe a structure independently of signatures?

Example: Countable atomless Boolean algebra

Let $\mathbb{B}$ be the countable atomless Boolean algebra.

Fact

For $a, a', b, b' \in \mathbb{B} \setminus \{0, 1\}$, if $a \leq b$ and $a' \leq b'$, then there is $\sigma \in \mathrm{Aut}(\mathbb{B})$ s.t. $\sigma(ab) = a'b'$.

Let B be the domain of $\mathbb{B}$, and its automorphism group $G \subseteq {}^{B}B$ is given as a permutation group.

1. There are exactly two fixed points $\{0, 1\}$ of the action.
2. Consider the action $G \curvearrowright B^2$.
3. Exactly one of the orbits (say, $<$) is an antireflexive binary relation.
4. The order can be recovered as $\Delta \cup < \cup (\{0\} \times B) \cup (B \times \{1\})$.

Ultrahomogeneity

Observation

It helps reconstruction to have many automorphisms.

Definition

A countable structure M is ultrahomogeneous if every isomorphism between finitely generated substructures of M extends to an automorphism of M.

Example

The countable atomless Boolean algebra is ultrahomogeneous.

Different variants of reconstruction

There are a few ways to interpret the LHS of

$$\mathrm{Aut}(M) \cong \mathrm{Aut}(N) \overset{?}{\Longleftrightarrow} M \cong N \tag{1}$$

1. Isomorphic as permutation groups
2. Isomorphic as topological groups
   Here $\mathrm{Aut}(M)$ has the pointwise convergence topology, i.e., the induced topology as a subset of ${}^{\omega}\omega$.
3. Isomorphic as abstract groups

Automorphism groups lose information as you go further down in the list.

Fact

If M and N are ω-categorical, then (1) holds as far as the isomorphism as topological groups is concerned.

Subgroups of small indices

Fact

For a countable structure M and closed subgroup $H \leq \mathrm{Aut}(M)$, the following are equivalent:

1. *H is open.*
2. $(G : H) \leq \aleph_0$.
3. $(G : H) < 2^{\aleph_0}$.

NB: all open subgroup of $\mathrm{Aut}(M)$ are closed.

Definition

A topological group has the small index property
$\iff$ every subgroup of index less than $2^{\aleph_0}$ is open.
A countable structure M has the s.i.p. $\iff$ $\mathrm{Aut}(M)$ does.

If $\mathrm{Aut}(M)$ has the small index property, its topology can be recovered from its abstract group structure only.

Which structures have the small index property?

Empirically, countable ultrahomogeneous structures fall into exactly one of the following classes:

1. Those known to have the small index property
(e.g., the countable atomless Boolean algebra; the countable universal ultrahomogeneous distributive lattice (Droste and Macpherson 2000))
2. Structures made up so they wouldn't have the small index property
3. Structures not known to have the small index property
(e.g., the universal ultrahomogeneous poset)

Fraïssé limit of finite Heyting algebras

The existence of the following follows from a general fact model theory and the amalgamation property of Heyting algebras (e.g., Maksimova 1977):

Definition

The Fraïssé limit L of finite Heyting algebras is the unique countable ultrahomogeneous structure into which every finite nontrivial Heyting algebra embeds.

Informally it lies halfway between the universal ultrahomogeneous poset and the countable atomless Boolean algebra.

Heyting algebras

Definition

A Heyting algebras A is a structure in the language $\{0, 1, \wedge, \vee, \rightarrow\}$ that expand bounded distributed lattices in which $a, b \in A$:

$$a \rightarrow b = \max\{c \in L \mid a \wedge c \leq b\}.$$

Example

Given a poset P, the lattice of upward closed subsets of P is (or can be made into) a Heyting algebra.

The category of Heyting algebras is known to be dually equivalent to the category of Stone spaces expanded with certain partial orders (Esakia spaces).

Main result

Theorem

The Fraïssé limit L of finite Heyting algebras has the small index property.

General method

Definition

A Polish group G has ample generic elements
$\iff$ for each $n < \omega$ the action of G on G^n defined by

$$g \cdot (\ldots, h, \ldots) = (\ldots, ghg^{-1}, \ldots)$$

has a comeager orbit.

Theorem (Kechris and Rosendal 2006)

If $\mathrm{Aut}(M)$ *has ample generics, then it has the strong index property*

It appears that if M has a "nontrivial" order, this method cannot be used.

Our method I

Instead, we adapted the "glueing" argument by Truss 1989 establishing the small index property of the countable atomless Boolean algebra.

Theorem (Maksimova 1977)

The class of nontrivial finite Heyting algebras has the superamalgamation property: Given finite Heyting algebras A_0, A_1, A_2 with $A_0 \subseteq (A_1 \cup A_2)$, there exists a finite Heyting algebra A containing $A_1 \cup A_2$ such that for $(i, j) = (1, 2), (2, 1)$

$$\forall(a_i \in A_i)\forall(a_j \in A_j)[a_i \leq a_j \implies \exists a_0 \in A_0\; a_i \leq a_0 \leq a_j]$$

Theorem (Yamamoto 202?)

$\mathrm{Aut}(L)$ *is simple.*

Our method II

Our argument also uses the following fact
("Aut(L) $\curvearrowright$ L is transitive enough"):

Lemma

Let X be the Esakia space of L, and consider the induced action of G on X. Let $Y \subseteq X$ be nonempty and finite. Then the G-orbit $G \cdot Y$ of Y in the natural action $G \curvearrowright \mathscr{P}(X)$ is of cardinality continuum.

Strengthening

Theorem

For $H \subseteq \mathrm{Aut}(L)$ *of a countable index, there exists a finite set* $A \subseteq L$ *such that* H *lies between the pointwise and setwise stabilizer of* A *("*L *has the strong small index property").*

Application I

"Definable" means first-order definable with parameters.

Definition (e.g. Casanovas and Farré 2004)

For a structure M:

1. M weakly eliminates imaginaries if for every definable relation R there exists a finite set $A \subseteq M$ such that every automorphism of M fixing A setwise fixes R setwise.
2. M eliminates imaginaries if for every definable relation R there exists a finite tuple $\bar{a} \subseteq M$ such that every automorphism of M fixing $\bar{a}$ pointwise fixes R setwise.

Such structures "have their definable quotients in them already."

Application II

Corollary

L weakly eliminates imaginaries, but it does not eliminate imaginaries.

Proof.

If a structure has the strong small index property, then it weakly eliminates imaginaries. OTOH, L does not have "codes for finite sets." □

Casanovas, E. and R. Farré (Mar. 2004). "Weak Forms of Elimination of Imaginaries". In: *Mathematical Logic Quarterly* 50, pp. 126–140.

Droste, M. and D Macpherson (2000). "The Automorphism Group of the Universal Distributive Lattice". In: *Algebra Universalis* 43, pp. 295–306.

Kechris, A. and C. Rosendal (2006). "Turbulance, Amalgamation, and Generic Automorphisms of Homogeneous Structures". In: *Proceedings of the London Mathematical Society*.

Maksimova, L. L. (Nov. 1977). "Craig's Theorem in Superintuitionistic Logics and Amalgamable Varieties of pseudo-Boolean Algebras". In: *Algebra and Logic* 16.6, pp. 427–455.

Truss, J. K. (1989). "Infinite Permutation Groups II. Subgroups of Small Index". In: *Journal of Algebra* 120.2, pp. 494 –515. ISSN: 0021-8693.

Yamamoto, K. (202?). "The Automorphism group of the Fraïssé Limit of Heyting Algebras". In: *The Journal of Symbolic Logic*.