Regular Categories and Soft Sheaf Representations

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joint work with Marco Abbadini

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  - 1970s: universal algebras (Comer, Cornish, Davey, Keimel, Wolf, ...)

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$\rightsquigarrow$ Frames of commuting congruences of an algebra A yield sheaf representations of A.

- **Characterisation** (Gehrke & van Gool, JPAA 2018)

Sheaves as “continuous presheaves”:

$$F\colon \Omega(X)^{\mathrm{op}} \to \mathsf{Set}$$

s.t. for all $S \subseteq \Omega(X)$ closed under binary $\wedge$, $F(\bigvee S) \cong \lim_{U \in S} F(U)$

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Theorem (Gehrke & van Gool, JPAA 2018)

Let X be a stably compact space and $A \neq \emptyset$ an algebra in a variety $\mathcal{V}$. There is a bijection between:

- *(isomorphism classes of) soft sheaf representations of A over X;*
- $(\wedge, \bigvee)$*-preserving maps* $\mathcal{K}(X)^{\mathrm{op}} \to \mathrm{Con}(A)$ *whose images consist of commuting congruences.*

$\mathcal{K}$-sheaves

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If P is any poset, this induces a functor

$$\begin{aligned} [P^{\mathrm{op}}, \operatorname{RegEpi} A] &\xrightarrow{\gamma_*} [P^{\mathrm{op}}, \mathsf{C}] \\ H &\longmapsto \gamma \circ H. \end{aligned}$$

Definition

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(K1) $F(\bot)$ is a terminal object of C.

(K2) For all $p, q \in P$, the following is a pullback square in C:

$$\begin{tikzcd}
F(p \vee q) \arrow[r, "\upharpoonright_{p\vee q,p}"] \arrow[d, "\upharpoonright_{p\vee q,q}"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(p) \arrow[d, "\upharpoonright_{p,p\wedge q}"] \\
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A $\mathcal{K}$-sheaf $F\colon P^{\mathrm{op}} \to \mathsf{C}$ is **soft** if, for all $p \in P$, $F(\top) \to F(p)$ is a regular epi.

Theorem

$\forall A \in \mathsf{C}$, *the functor* $\gamma_* \colon [P^{\mathrm{op}}, \mathsf{RegEpi}\, A] \to [P^{\mathrm{op}}, \mathsf{C}]$ *induces an isomorphism*

$$\mathsf{M} \cong \text{s-Sh}_{\mathcal{K}}^{A}(P, \mathsf{C})$$

where

- M *is the full subcategory of* $[P^{\mathrm{op}}, \mathsf{RegEpi}\, A]$ *on the* $(\wedge, \bigvee)$*-preserving maps whose images consist of* **ker-commuting** *elements;*
- $\text{s-Sh}_{\mathcal{K}}^{A}(P, \mathsf{C})$ *is the category of soft* $\mathcal{K}$*-sheaf representations of* A *over* P*.*

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Rmk: If C is Barr-exact and we take **poset reflections**, $\mathsf{RegEpi}\, A$ can be replaced with $\mathsf{Equiv}\, A$. In that case, the ker-commuting regular epis correspond to the commuting equivalence relations.

Ω-sheaves

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$$\begin{tikzcd} L^{\mathrm{op}} \arrow[r, tail, "\lambda"] & \mathrm{Filt}(L) & \sigma\mathrm{Filt}(L) \arrow[l, tail, "\kappa"'] \end{tikzcd}$$

where $\sigma\mathrm{Filt}(L)$ is the domain of **Scott-open filters** of L

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{[L^{\mathrm{op}}, \mathsf{C}]} \arrow[r, bend right, "\mathsf{Lan}_\lambda"'] & {[\mathrm{Filt}(L), \mathsf{C}]} \arrow[l, "\lambda^*"', "\top"] \arrow[r, "\kappa^*"] & {[\sigma\mathrm{Filt}(L), \mathsf{C}].}
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Proposition

The previous adjunction restricts to an equivalence

$$\boxed{\omega\text{-}\lim[L^{\mathrm{op}}, \mathsf{C}] \simeq \omega\text{-}\mathrm{colim}[\sigma\mathrm{Filt}(L), \mathsf{C}]}\,. \qquad (⛰)$$

Definition

An **Ω-sheaf** on a complete lattice P is a functor $F\colon P^{\mathrm{op}} \to \mathsf{C}$ such that:

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Theorem

If L is a stably continuous lattice, (⛰) restricts to a fully faithful functor

$$\mathsf{Sh}_{\mathcal{K}}(\sigma\mathrm{Filt}(L)^{\mathrm{op}}, \mathsf{C}) \hookrightarrow \mathsf{Sh}_{\Omega}(L, \mathsf{C}).$$

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Rmk: If $L = \Omega(X)$ with X stably compact, then we can replace $\sigma\mathrm{Filt}(L)^{\mathrm{op}}$ with $\mathcal{K}(X)$ in the previous theorem.

This restricts to sheaf representations: for all $A \in \mathsf{C}$,

$$\mathsf{M} \cong \mathrm{s\text{-}Sh}^{A}_{\mathcal{K}}(\sigma\mathrm{Filt}(L)^{\mathrm{op}}, \mathsf{C}) \hookrightarrow \mathrm{s\text{-}Sh}^{A}_{\Omega}(L, \mathsf{C}) \tag{$\blacktriangle$}$$

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where an Ω-sheaf $F \colon L^{\mathrm{op}} \to \mathsf{C}$ is **soft** if, $\forall k \in \sigma\mathrm{Filt}(L)$, $F(\top) \twoheadrightarrow \mathrm{colim}_{x \in k} F(x)$

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As finitary varieties are bicomplete Barr-exact categories in which directed colimits commute with finite limits, Gehrke & van Gool's result follows by taking **poset reflections** in (◭).

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Thank you for your attention!