



Regular Categories and Soft Sheaf Representations

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joint work with Marco Abbadini

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- ▶ **Characterisation** (Gehrke & van Gool, JPAA 2018)

Sheaves as “continuous presheaves”:

$$F : \Omega(X)^{\text{op}} \rightarrow \text{Set}$$

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Theorem (Gehrke & van Gool, JPAA 2018)

Let X be a stably compact space and $A \neq \emptyset$ an algebra in a variety \mathcal{V} . There is a bijection between:

- ▶ (isomorphism classes of) soft sheaf representations of A over X ;
- ▶ (\wedge, \bigvee) -preserving maps $\mathcal{K}(X)^{\text{op}} \rightarrow \text{Con}(A)$ whose images consist of commuting congruences.

\mathcal{K} -sheaves

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▶ For all $A \in C$, $(\text{Quo } A)^{\text{op}} \xrightarrow{\text{ker}} \text{Equiv } A$ is an order-embedding.

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If P is any poset, this induces a functor $[P^{\text{op}}, \text{RegEpi } A] \xrightarrow{\gamma_*} [P^{\text{op}}, C]$
 $H \longmapsto \gamma \circ H.$

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(K1) $F(\perp)$ is a terminal object of \mathbf{C} .

(K2) For all $p, q \in P$, the following is a pullback square in \mathbf{C} :

$$\begin{array}{ccc} F(p \vee q) & \xrightarrow{\uparrow_{p \vee q, p}} & F(p) \\ \uparrow_{p \vee q, q} \downarrow & \lrcorner & \downarrow \uparrow_{p, p \wedge q} \\ F(q) & \xrightarrow{\uparrow_{q, p \wedge q}} & F(p \wedge q) \end{array}$$

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(K1) $F(\perp)$ is a **subterminal** object of \mathbf{C} .

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A \mathcal{K} -sheaf $F: P^{\text{op}} \rightarrow \mathbf{C}$ is **soft** if, for all $p \in P$, $F(\top) \rightarrow F(p)$ is a regular epi.

Theorem

$\forall A \in \mathcal{C}$, the functor $\gamma_*: [P^{\text{op}}, \text{RegEpi } A] \rightarrow [P^{\text{op}}, \mathcal{C}]$ induces an isomorphism

$$\mathcal{M} \cong \text{s-Sh}_{\mathcal{K}}^A(P, \mathcal{C})$$

where

- ▶ \mathcal{M} is the full subcategory of $[P^{\text{op}}, \text{RegEpi } A]$ on the (\wedge, \vee) -preserving maps whose images consist of **ker-commuting** elements;
- ▶ $\text{s-Sh}_{\mathcal{K}}^A(P, \mathcal{C})$ is the category of soft \mathcal{K} -sheaf representations of A over P .

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Rmk: If \mathcal{C} is Barr-exact and we take **poset reflections**, $\text{RegEpi } A$ can be replaced with $\text{Equiv } A$. In that case, the ker-commuting regular epis correspond to the commuting equivalence relations.

Ω -sheaves

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where $\sigma\text{Filt}(L)$ is the domain of **Scott-open filters** of L

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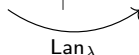
$$[L^{\text{op}}, \mathcal{C}] \xleftarrow{\lambda^*} [\text{Filt}(L), \mathcal{C}] \xrightarrow{\kappa^*} [\sigma\text{Filt}(L), \mathcal{C}].$$

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Proposition

The previous adjunction restricts to an equivalence

$$\omega\text{-lim}[L^{\text{op}}, \mathcal{C}] \simeq \omega\text{-colim}[\sigma\text{Filt}(L), \mathcal{C}].$$



Definition

An **Ω -sheaf** on a complete lattice P is a functor $F: P^{\text{op}} \rightarrow \mathbf{C}$ such that:

$$(\Omega 1) = (K1)$$

$$(\Omega 2) = (K2)$$

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Theorem

If L is a stably continuous lattice, (\blacktriangle) restricts to a fully faithful functor

$$\text{Sh}_{\mathcal{K}}(\sigma\text{Filt}(L)^{\text{op}}, \mathbf{C}) \hookrightarrow \text{Sh}_{\Omega}(L, \mathbf{C}).$$

This is an equivalence if *directed colimits in \mathbf{C} commute with finite limits*.

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Rmk: If $L = \Omega(X)$ with X stably compact, then we can replace $\sigma\text{Filt}(L)^{\text{op}}$ with $\mathcal{K}(X)$ in the previous theorem.

This restricts to sheaf representations : for all $A \in \mathcal{C}$,

$$\mathbf{M} \cong \mathbf{s}\text{-Sh}_{\mathcal{K}}^A(\sigma\text{Filt}(L)^{\text{op}}, \mathcal{C}) \hookrightarrow \mathbf{s}\text{-Sh}_{\Omega}^A(L, \mathcal{C}) \quad (\blacktriangle)$$

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where an Ω -sheaf $F: L^{\text{op}} \rightarrow \mathcal{C}$ is **soft** if, $\forall k \in \sigma\text{Filt}(L)$, $F(\top) \rightarrow \text{colim}_{x \in k} F(x)$

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Rmk (sheaves vs Ω -sheaves): For any space X , a presheaf $F: \Omega(X)^{\text{op}} \rightarrow \mathbf{C}$ is a sheaf iff it is an Ω -sheaf and $F(\emptyset)$ is terminal.

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As finitary varieties are bicomplete Barr-exact categories in which directed colimits commute with finite limits, Gehrke & van Gool's result follows by taking **poset reflections** in (\blacktriangle) .

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Thank you for your attention!

