The coordinatization of the spectra of ℓ-groups and vector lattices

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joint work with L. Carai and S. Lapenta

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Main ideas

- Study the dual side of ℓ-groups and Riesz spaces introduced in the previous talk.
- Compare the functors in the duality with $\mathtt{Spec}$, $\mathsf{PWL}_{\mathbb{Z}}$ and $\mathsf{PWL}_{\mathbb{R}}$.
- Application 1: a concrete representation of $\mathtt{Spec}$ into ultrapowers of $\mathbb{R}$.
- Application 2: an alternative proof of Panti's characterisation of prime ideals.

Recap from the previous talk

The Galois connection

- V indicates either the variety of abelian ℓ-groups or the variety of vector lattices.
- PWL indicates either the piecewise homogeneous functions with coefficients in $\mathbb{R}$ or in $\mathbb{Z}$.
- $\mathcal{U}$ always denotes some ultrapower of $\mathbb{R}$ in V.

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For cardinals $\kappa < \gamma$, the operators $\mathbb{V}$ and $\mathbb{I}$ are defined, for any $T \subseteq \mathscr{F}_\kappa$ and $S \subseteq \mathcal{U}^\kappa$,

$$
\begin{aligned}
{}^{\kappa}\mathbb{V}^{\gamma}_{\mathcal{U}}(T) &:= \{x \in \mathcal{U}^\kappa \mid t(x) = 0 \text{ for all } t \in T\} \\
{}^{\kappa}\mathbb{I}^{\gamma}_{\mathcal{U}}(S) &:= \{t \in \mathscr{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\}.
\end{aligned}
$$

They form a Galois connection that extends to dual adjunction.

The duality

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Notice that for any $S \subseteq \mathcal{U}^\kappa$,

$$\mathbb{V}\,\mathbb{I}(S) = S \text{ if and only if } S = \mathbb{V}(T) \text{ for some } T \subseteq \mathscr{F}_\kappa$$

The operator $\mathbb{V}\,\mathbb{I}$

$\mathbb{V}\,\mathbb{I}$ is (almost) topological

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So we need to consider ${\mathcal{U}_{o}}^{\kappa} := \mathcal{U}^{\kappa} \setminus \{O\}$ and modify $\mathbb{V}$ accordingly: $\mathbb{V}_{o}(S) := \mathbb{V}(S) \setminus \{O\}$.

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The Zariski topology on $\mathcal{U}_o^\kappa$ is not even T_0. Indeed, $t(x) = 0$ implies $t(x + x) = t(x) + t(x)$. Whence x and $2x$ cannot be separated by an open set.

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Therefore, we will consider the T_0-reflection of $\mathcal{U}_o^\kappa$. This is equivalently obtained by taking a quotient over the relation

$$x \sim y \text{ if and only if } \mathbb{V}\,\mathbb{I}(x) = \mathbb{V}\,\mathbb{I}(y).$$

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This is because all closed subsets of $\mathcal{U}_o^\kappa$ are saturated w.r.t. the relation $\sim$.

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The other direction is a consequence of the fact that finitely generated ideals are principal in $\mathbb{V}$.

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Indeed, notice that being irreducible means to be join-prime in the lattice of closed sets. The latter is order-dual to the lattice of ideals, in which prime ideals are exactly the meet-prime elements.

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It is T_0 by construction, and by taking the quotient the compact open sets and the irreducible closed ones do not change.

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Corollary

For any κ*-generated object in* $A \in \mathbb{V}$ *there exists an embedding of* $\mathtt{Spec}\, A$ *into* $\mathcal{U}_o^\kappa$ *such that* $A \cong {}^*\mathrm{PWL}(\mathtt{Spec}(A))$*.*

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Corollary

A topological space is the spectrum of some $A \in \mathbb{V}$ iff it is a closed subspace of some $\mathcal{U}_o^\kappa/\!\sim$.

Irreducible closed

Non standard tools

If $\mathcal{U} = \prod \mathbb{R}/\mathcal{F}$ for some ultrafilter $\mathcal{F}$, every subset $X \subseteq \mathbb{R}^n$ can be associated with a subset ${}^*X$ of $\mathcal{U}^n$ defined as

$$\{x \in \mathcal{U}^n \mid \{i \in I \mid \pi_i(x) \in X\} \in \mathcal{F}\}$$

and called the enlargement of X. Similarly, every predicate $P \subseteq \mathbb{R}^n$ and function $f : \mathbb{R}^n \to \mathbb{R}$ can be enlarged to ${}^*P \subseteq \mathcal{U}^n$ and ${}^*f : \mathcal{U}^n \to \mathcal{U}$.

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Transfer principle (Łoś Theorem)

Let φ be a first order formula and ${}^*\varphi$ the formula obtained by replacing every predicate symbol P and every function symbol f with ${}^*P$ and ${}^*f$. Then φ is true in $\mathbb{R}$ iff ${}^*\varphi$ is true in $\mathcal{U}$.

Orthogonal decomposition theorem (Goze 1995)

Any $x \in \mathcal{U}_o^n$ can be written in a unique way as

$$x = \alpha_1 v_1 + \cdots + \alpha_k v_k$$

where

1. $v_1, \ldots, v_k$ are orthonormal vectors of $\mathbb{R}^n$,
2. $0 < \alpha_1, \ldots, \alpha_k \in \mathcal{U}$, and
3. α_{i+1}/α_i is infinitesimal for every $i < k$.

Thus, each $x \in \mathcal{U}_o^n$ gets associated with a sequence $\mathbf{v} = (v_1, \dots, v_k)$ of orthonormal vectors, which we call index.

For an index $\mathbf{v}$, let $\mathrm{Hcone}(\mathbf{v})$ be the set of points of $\mathcal{U}_o^n$ whose index is a truncation of $\mathbf{v}$.

Theorem (Carai, Lapenta, and S.)

In the Zariski topology of $\mathcal{U}_o^n$ relative to vector lattices each irreducible closed of $\mathcal{U}_o^n$ is $\mathrm{Hcone}(\mathbf{v})$ for some index $\mathbf{v}$. In other words,

$$\mathbb{V}\,\mathbb{I}(\{x\}) = \mathrm{Hcone}(\mathbf{v}(x)).$$

Abelian ℓ-groups and $\mathbb{Z}$-reduced indices

Definition

If $w \in \mathbb{R}^n$, let $\langle w \rangle$ be the smallest subspace containing w that admits a basis in $\mathbb{Z}^n$.

An index $\mathbf{v} = (v_1, \ldots, v_k)$ is $\mathbb{Z}$-reduced if $\langle v_i \rangle$ and $\langle v_j \rangle$ are orthogonal for each $i \neq j$.

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Using a sort of Gram-Schmidt process, we can associate to each index $\mathbf{v}$ a unique $\mathbb{Z}$-reduced index $\mathrm{red}(\mathbf{v})$.

Abelian ℓ-groups and $\mathbb{Z}$-reduced indices

Theorem (Carai, Lapenta, and S.)

In the Zariski topology of $\mathcal{U}_o^n$ relative to abelian ℓ-groups each irreducible closed of $\mathcal{U}_o^n$ is of the form

$$\bigcup\{\mathrm{Hcone}(\mathbf{w}) \mid \mathrm{red}(\mathbf{w}) = \mathbf{v}\}.$$

for some $\mathbb{Z}$-reduced index $\mathbf{v}$.

Indices and cones

If $\mathbf{v}$ is an index, we say that a closed cone $C \subseteq \mathbb{R}^n$ is a $\mathbf{v}$-cone if there exist real numbers $r_2, \ldots, r_k > 0$ such that C is generated by $\{v_1, v_1 + r_2 v_2, \ldots, v_1 + r_2 v_2 + \cdots + r_k v_k\}$.

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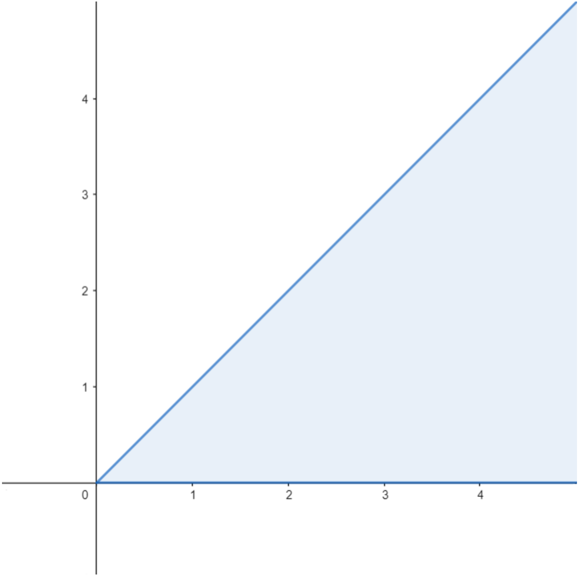
Proposition

Hcone($\mathbf{v}$) is the intersection of the enlargements of all the $\mathbf{v}$-cones.

$\mathbf{v} = ((1,0),(0,1)).$



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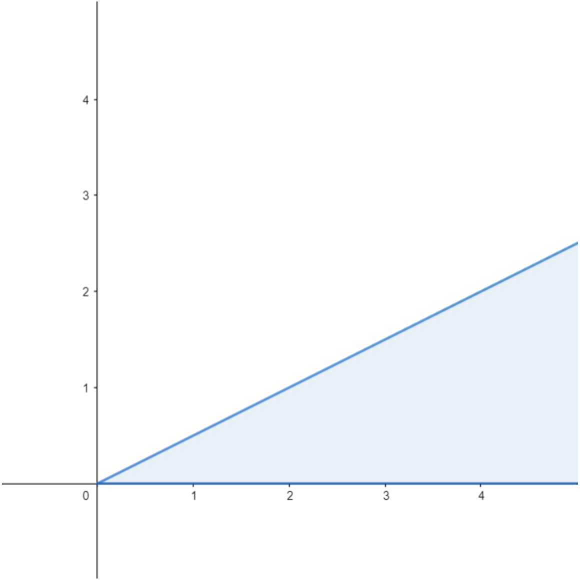

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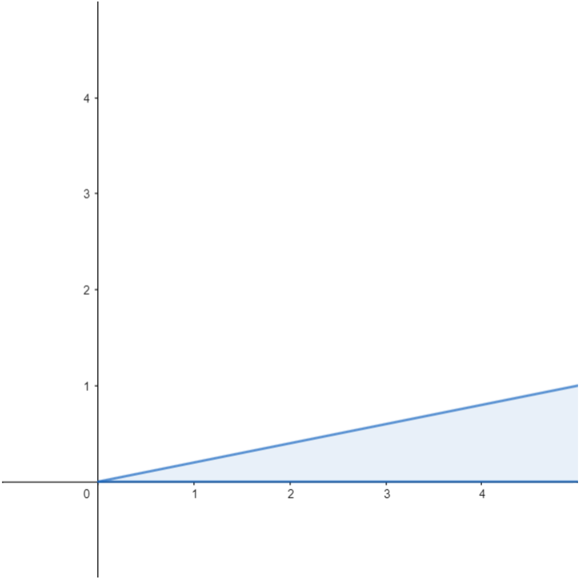

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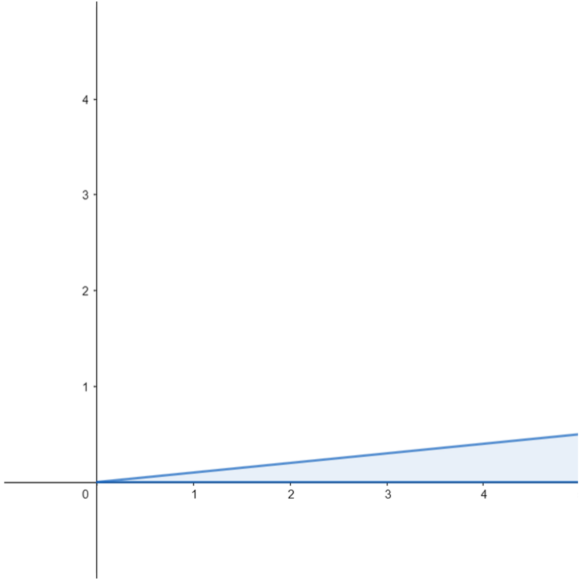

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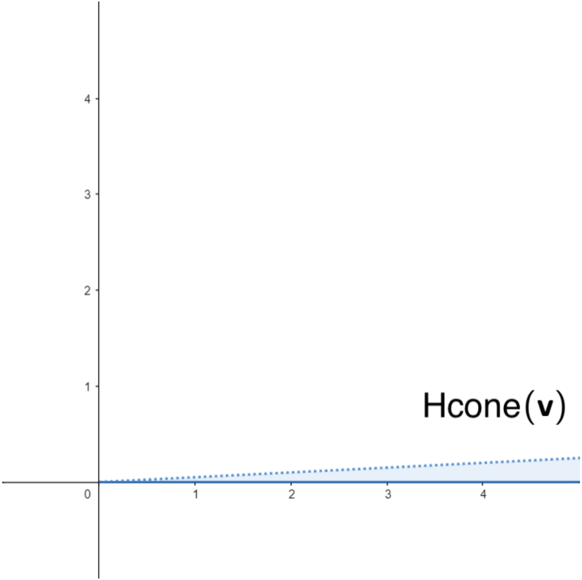

Primes and indices

Theorem (Carai, Lapenta, and S.)

For any $f \in \mathrm{PWL}(\mathbb{R}^n)$,

$^*f$ *vanishes on* $\mathrm{Hcone}(\mathbf{v})$ *iff* f *vanishes on some* $\mathbf{v}$*-cone.*

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In addition, if $f \in \mathrm{PWL}_{\mathbb{Z}}(\mathbb{R}^n)$ *and* v *is* $\mathbb{Z}$*-reduced, then*

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As a corollary, we obtain the description of prime ℓ-ideals in finitely generated vector lattices and abelian ℓ-groups due to Panti.

Theorem (Panti 1999)

Each prime ideal of $\mathscr{F}_n$ *is of the form*

$\{f \in \mathrm{PWL}(\mathbb{R}^n) \mid f$ *vanishes on a* $\mathbf{v}$*-cone*$\}$ *for a uniquely determined (reduced) index* $\mathbf{v}$*.*

Thank You!