Modal reduction principles across relational semantics

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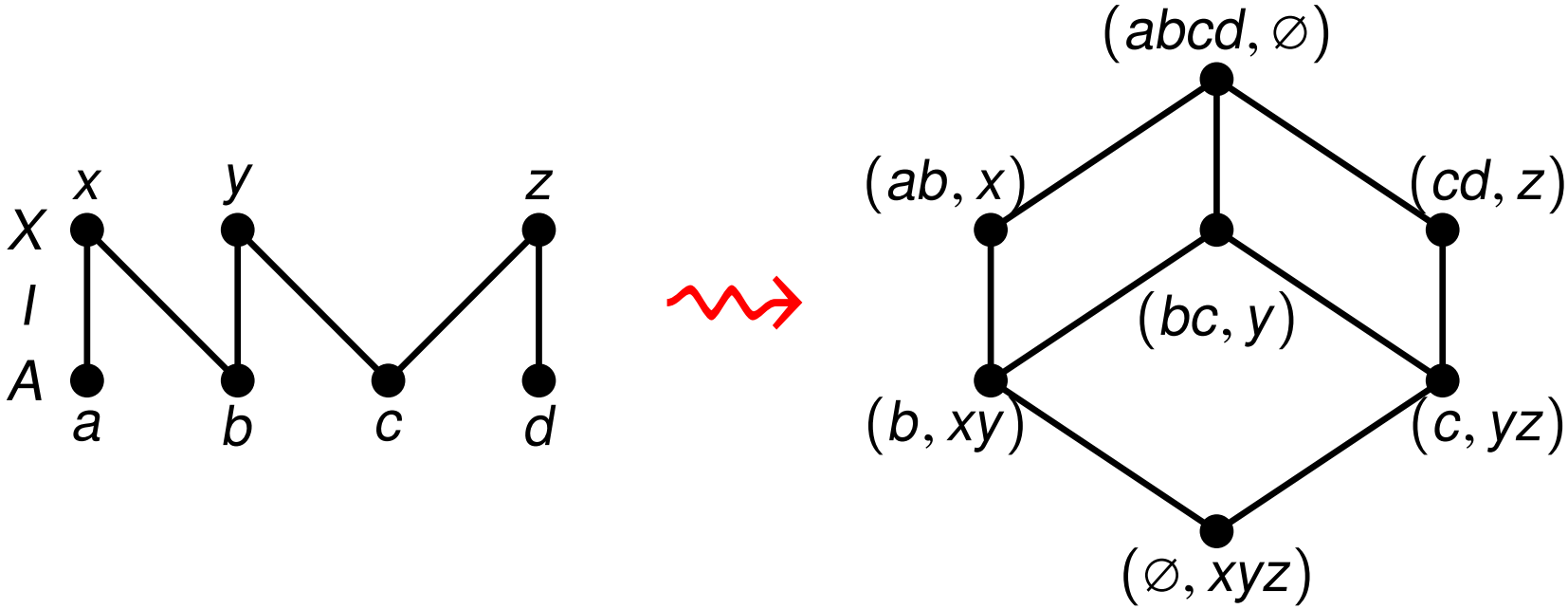
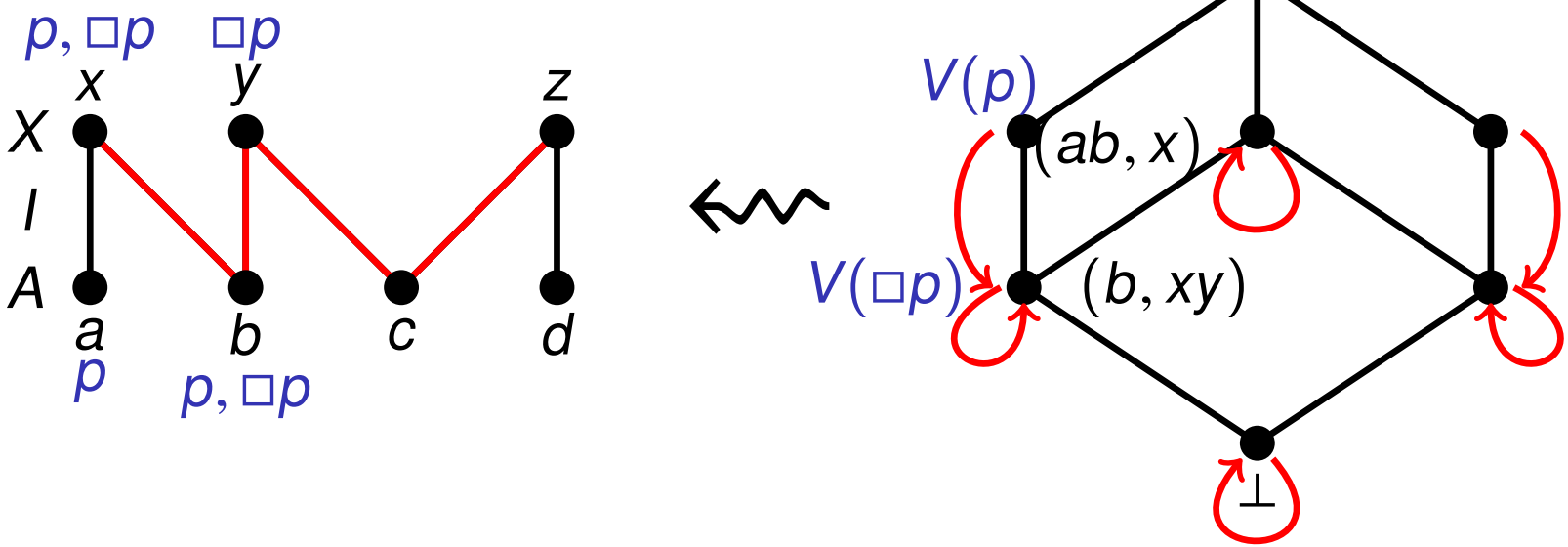
Polarities

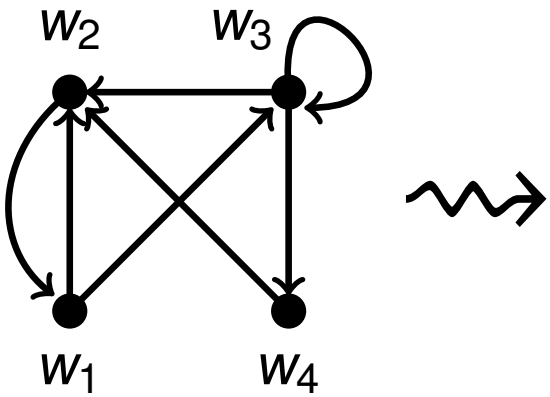


Polarity based semantics for LE-logics

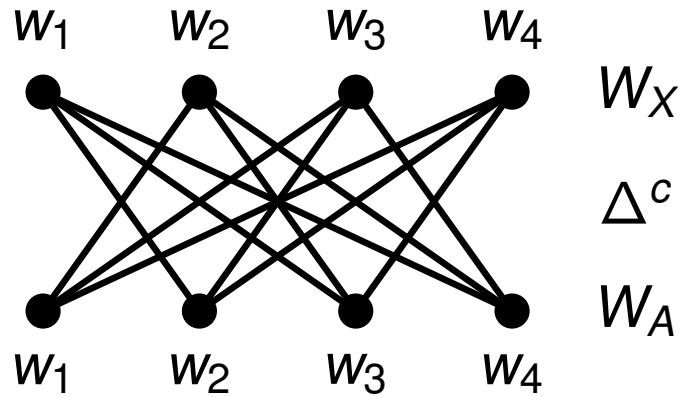


Lifting of a Kripke frame

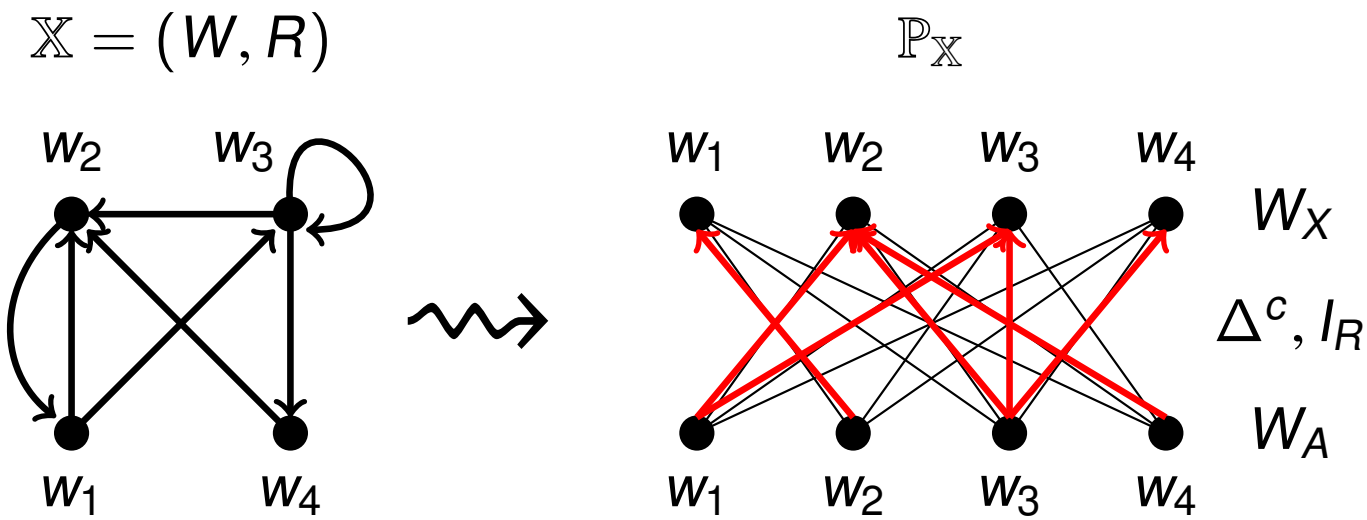
$\mathbb{X} = (W, R)$



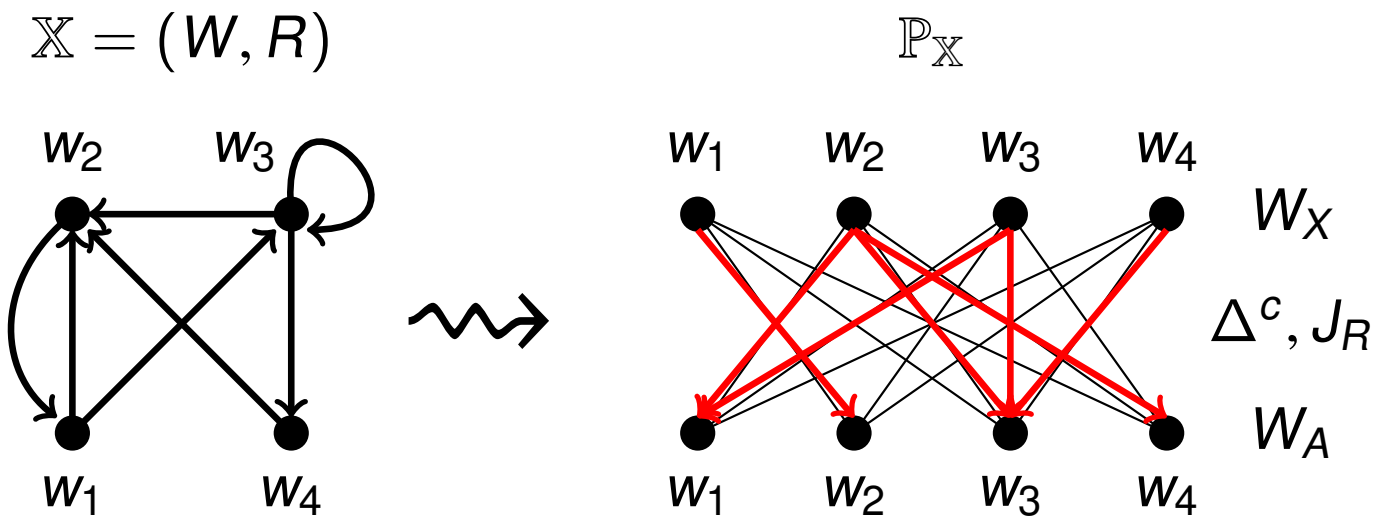
$\mathbb{P}_{\mathbb{X}}$



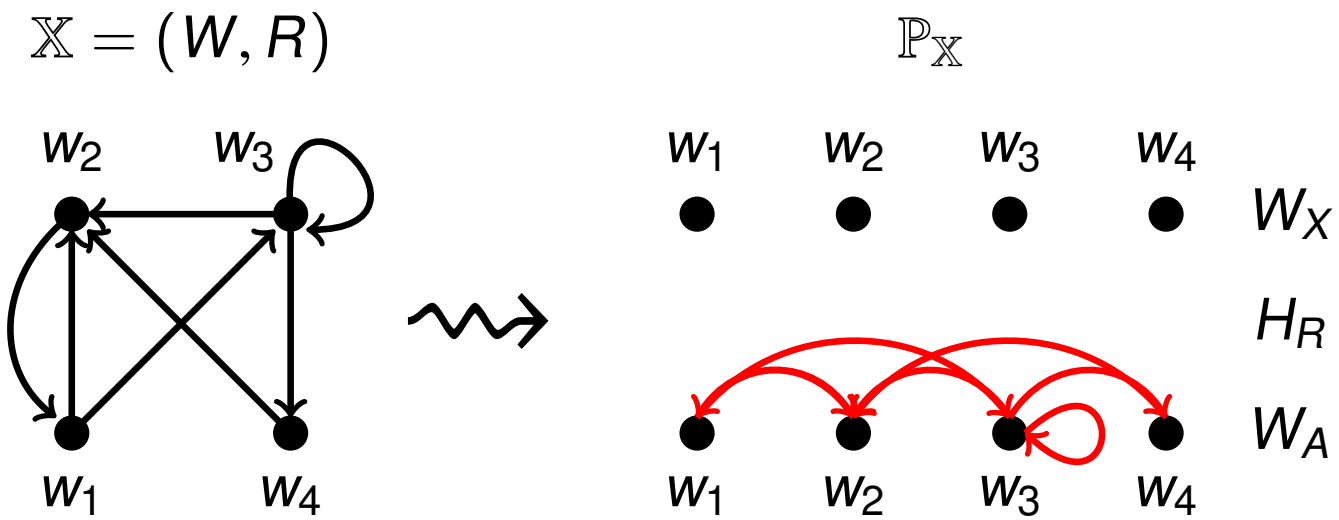
Lifting of a Kripke frame - I lifting



Lifting of a Kripke frame - J lifting



Lifting of a Kripke frame - H lifting



An example of FO-correspondents

$$
\begin{array}{ll}
 & \Box p \leq \Box\Box p \\
\text{iff} & \bigvee\{\mathbf{j} \mid \mathbf{j} \leq \Box p\} \leq \Box\Box \bigwedge\{\mathbf{m} \mid p \leq \mathbf{m}\} \\
\text{iff} & \bigvee\{\mathbf{j} \mid \mathbf{j} \leq \Box p\} \leq \bigwedge\{\Box\Box \mathbf{m} \mid p \leq \mathbf{m}\} \\
\text{iff} & \forall \mathbf{j} \forall \mathbf{m} \, (\mathbf{j} \leq \Box p \;\&\; p \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \Box\Box \mathbf{m}) \\
\text{iff} & \forall \mathbf{j} (\mathbf{j} \leq \Box \mathbf{m} \Rightarrow \mathbf{j} \leq \Box\Box \mathbf{m})
\end{array}
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$$

In Kripke frames, the interpretation becomes

$$\forall x \forall y (x \notin R_{\Box}^{-1}[y] \Rightarrow x \notin R_{\Box}^{-1}[R_{\Box}^{-1}[y]]),$$

$$\text{i.e., } R_{\Box} \circ R_{\Box} \subseteq R_{\Box}$$

In polarity based frames it becomes

$$\forall a \forall x (a^{\uparrow\downarrow} \subseteq R_{\Box}^{(0)}[x^{\downarrow\uparrow}] \Rightarrow a^{\uparrow\downarrow} \subseteq R_{\Box}^{(0)}[I^{(1)}[R_{\Box}^{[0]}[x^{\downarrow\uparrow}]]]),$$

$$\text{i.e., } R_{\Box}^{(0)}[\cdot] \subseteq R_{\Box}^{(0)}[I^{(1)}[R_{\Box}^{[0]}[\cdot]]]$$

A more complicated example

$$
\begin{array}{ll}
 & \Box\Diamond p \leq \Box\Diamond\Diamond p \\
\text{iff} & \forall \mathbf{j}\forall \mathbf{m}(\mathbf{j} \leq \Box\Diamond p \;\&\; \Diamond\Diamond p \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \Box \mathbf{m}) \\
\text{iff} & \forall \mathbf{j}\forall \mathbf{m}(\mathbf{j} \leq \Box\Diamond\blacksquare\blacksquare \mathbf{m} \Rightarrow \mathbf{j} \leq \Box \mathbf{m}) \\
\text{iff} & \forall \mathbf{m}(\Box\Diamond\blacksquare\blacksquare \mathbf{m} \leq \Box \mathbf{m})
\end{array}
$$

A more complicated example

$$\Box\Diamond p \leq \Box\Diamond\Diamond p$$

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iff $\forall \mathbf{j}\forall \mathbf{m}(\mathbf{j} \leq \Box\Diamond\blacksquare\blacksquare\mathbf{m} \Rightarrow \mathbf{j} \leq \Box\mathbf{m})$

iff $\forall \mathbf{m}(\Box\Diamond\blacksquare\blacksquare\mathbf{m} \leq \Box\mathbf{m})$

In polarities it is

$$R_{\Box}^{(0)}[R_{\Diamond}^{(0)}[R_{\blacksquare}^{(0)}[I^{(1)}[R_{\blacksquare}^{(0)}[\cdot]]]]] \subseteq R_{\Box}^{(0)}[\cdot]$$

i.e. $R_{\Box}; R_{\Diamond}; (R_{\blacksquare} ;_I R_{\blacksquare}) \subseteq R_{\Box}$

A more complicated example

$$\Box\Diamond p \leq \Box\Diamond\Diamond p$$
iff $\forall \mathbf{j}\forall \mathbf{m}(\mathbf{j} \leq \Box\Diamond p \;\&\; \Diamond\Diamond p \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \Box\mathbf{m})$
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In Kripke frames it becomes

$$R_{\Box} \subseteq R_{\Box} \star R_{\Diamond} \star (R_{\blacksquare} \circ R_{\blacksquare}),$$

A more complicated example

$$\begin{aligned} & \Box\Diamond p \leq \Box\Diamond\Diamond p \\ \text{iff} \quad & \forall \mathbf{j}\forall \mathbf{m}(\mathbf{j} \leq \Box\Diamond p \;\&\; \Diamond\Diamond p \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \Box \mathbf{m}) \\ \text{iff} \quad & \forall \mathbf{j}\forall \mathbf{m}(\mathbf{j} \leq \Box\Diamond\blacksquare\blacksquare \mathbf{m} \Rightarrow \mathbf{j} \leq \Box \mathbf{m}) \\ \text{iff} \quad & \forall \mathbf{m}(\Box\Diamond\blacksquare\blacksquare \mathbf{m} \leq \Box \mathbf{m}) \end{aligned}$$

In polarities it is

$$\begin{aligned} & R_\Box^{(0)}[R_\Diamond^{(0)}[R_\blacksquare^{(0)}[I^{(1)}[R_\blacksquare^{(0)}[\cdot]]]]] \subseteq R_\Box^{(0)}[\cdot] \\ \text{i.e.} \quad & R_\Box ; R_\Diamond ; (R_\blacksquare ;_I R_\blacksquare) \subseteq R_\Box \end{aligned}$$

In Kripke frames it becomes

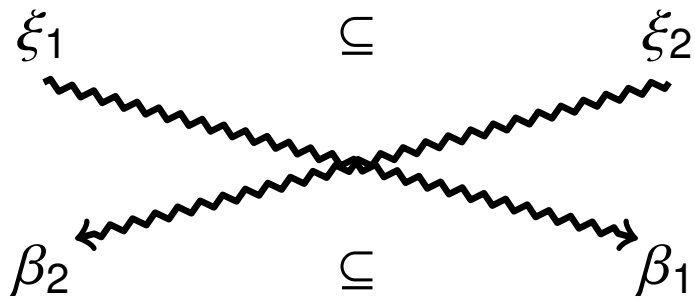
$$R_\Box \subseteq R_\Box \star R_\Diamond \star (R_\blacksquare \circ R_\blacksquare),$$

where $(R \star S)[\cdot] = ((R^c)^{(0)}[(S^c)^{(0)}[\cdot])^c$

More in general

In order to lift from Kripke to polarities,

$$
\begin{array}{lcll}
\star & \rightsquigarrow & ; & \text{taking types into account} \\
\circ & \rightsquigarrow & ;_I & \text{taking types into account} \\
\Delta & \rightsquigarrow & I \text{ or } J = I^c & \text{depending on context}
\end{array}
$$



Some examples

Reflexivity: $\Delta \subseteq R_{\Box} \quad \rightsquigarrow \quad R_{\Box} \subseteq I$

Symmetry: $R_{\blacklozenge} \subseteq R_{\Diamond} \quad \rightsquigarrow \quad R_{\Diamond} \subseteq R_{\blacklozenge}$

Transitivity: $R_{\Box} \circ R_{\Box} \subseteq R_{\Box} \quad \rightsquigarrow \quad R_{\Box} \subseteq R_{\Box} ;_I R_{\Box}$

Reflexivity: $R_{\Diamond} \subseteq R_{\blacklozenge} \quad \rightsquigarrow \quad R_{\blacklozenge} \subseteq R_{\Diamond}$

Conclusions

- Intuitive translation of mrps' correspondents
- Towards parametric correspondence:
  - Kripke frames *lift* to polarities changing Δ to I and J
  - Polarities *lift* to MV-polarities changing **2** to **H**
  - Kripke frames *shift* to graph-based frames, $\Delta \rightsquigarrow E$
  - Is it possible to transfer other results parametrically?
  - Is it possible to extend beyond mrps?

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Thank you!