Algebraizability as an algebraic structure

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Plan:

1. Categories of logics and translations
2. Remote algebraizability
3. More on categories of logics and translations
4. Algebraizability as an algebraic structure
5. Further points

Categories of logics and translations

Signature : $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$

$\rightsquigarrow \mathrm{Fm}_\Sigma(X)$ formulas

Strict morphism $\Sigma \xrightarrow{f} \Sigma'$: $(f_n : \Sigma_n \to \Sigma'_n)_{n \in \mathbb{N}}$

Flexible morphism : $(f_n : \Sigma_n \longrightarrow \mathrm{Fm}_{\Sigma'}(X)[n])_{n \in \mathbb{N}}$

$\rightsquigarrow$ Categories $\mathrm{Sig}^{\mathrm{strict}}$, $\mathrm{Sig}^{\mathrm{flex}}$

$\cong$

$\mathrm{Set}^{\mathbb{N}}$

Categories of logics and translations

Logic: $L = (\Sigma, \vdash)$

signature

$\boxed{\text{finitary, structural cons. rel. on } Fm_\Sigma(X)}$

these form a complete lattice in $\mathcal{P}(\mathcal{P}(Fm_\Sigma(X)) \times Fm_\Sigma(X))$

$\Rightarrow$
- can generate a cons. rel. from rules
- $$Fm_\Sigma(X) \xrightarrow{f} Fm_{\Sigma'}(X)$$
  $$f^*(\vdash') \longleftarrow \vdash'$$
  $$\vdash \longmapsto f_*(\vdash)$$

Categories of logics and translations

Strict / flexible translation $(\Sigma, \vdash) \longrightarrow (\Sigma', \vdash')$:

strict / flex. signature morph. $\Sigma \xrightarrow{f} \Sigma'$

s.t. $\Gamma \vdash \varphi \Rightarrow f(\Gamma) \vdash' f(\varphi)$

Equivalent: $\vdash \subseteq f^*(\vdash')$

$f_*(\vdash) \subseteq \vdash'$

$\rightsquigarrow$ categories $\mathrm{Log}^{\mathrm{strict}}$, $\mathrm{Log}^{\mathrm{flex}}$

Categories of logics and translations

$$\begin{array}{c} \mathrm{Log}^{\mathrm{strict}/\mathrm{flex}} \\ \big\downarrow \\ \mathrm{Sig}^{\mathrm{strict}/\mathrm{flex}} \end{array}$$

$\leftarrow$ category fibered in complete lattices over Sig

$\Rightarrow$ Log has all (co)limits that exist in Sig

$\mathrm{Log}^{\mathrm{strict}}$ (co)complete

$\mathrm{Log}^{\mathrm{flex}}$ not

E.g. $L_1 \times L_2 = \ldots$

$L_1 \amalg L_2 = \langle \Sigma_1 \amalg \Sigma_2 \mid \mathrm{rules}_1, \mathrm{rules}_2 \rangle$

Categories of logics and translations

For $L \overset{f}{\underset{g}{\rightrightarrows}} L'$ define

$$f \sim g \;:\Longleftrightarrow\; \forall \varphi \in L: \; f(\varphi) \dashv\vdash' g(\varphi)$$

Category HoLog:

Objects: logics

Morphisms: $\mathrm{HoLog}(L, L') := \mathrm{Log}^{\mathrm{flex}}(L, L')/\sim$

Fact: HoLog is complete & cocomplete.

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Possible Translations Semantics (Carnielli):

$\{L \xrightarrow{f_i} L_i \mid i \in I\}$ s.t. $\Gamma \vdash \varphi \iff \forall i : f_i(\Gamma) \vdash_i f_i(\varphi)$

Equivalently: $\vdash \; = \bigwedge_{i \in I} f_i^*(\vdash_i)$

Equivalently: $L \xrightarrow{\langle f_i \rangle} \prod L_i$ is conservative
(i.e. reflects deductions)

Remote Algebraization

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L is remotely algebraizable (protoalgebraic, equivalential, ..)
if the L_i can be chosen algebraizable.

$L = (\Sigma, \vdash)$ is protoalgebraic $:\Longleftrightarrow$

$\exists$ set of binary connectives $\Rightarrow = \{\to_1, \dots, \to_n\}$

s.t. $\emptyset \vdash x \to_i x$ and $\{x, x \to_1 y, \dots, x \to_n y\} \vdash y$

L is n-protoalgebraic if $|\Rightarrow| \leq n$.

Generic n-protoalgebraic logic:

$$P_n := (\{\to_1, \dots, \to_n\}, \vdash)$$

generated by those rules

[L is n-protoalgebraic iff $\exists\ P_n \to L$]

Remote Algebraization

Protoalgebraic logics are not closed under products:

$\prod_{n\in\mathbb{N}} P_n$ does not satisfy Modus ponens (finitary)

... but under finite products.

L remotely protoalgebraic via finite family

$\iff$ $\exists$ conservative translation $L \longrightarrow L'$

an n-protoalg. logic

$\iff$ coproduct inclusion $L \longrightarrow L \amalg P_n$ is conservative

$$\begin{tikzcd}
L \arrow[r] \arrow[dr] & L \amalg P_n \arrow[d, dashed] & P_n \arrow[l] \arrow[dl] \\
 & L' &
\end{tikzcd}$$

Prop.: $L \longrightarrow L \amalg P_n$ is conservative iff

1. L has theorems

or 2. L has no rules of the form $\{x\} \vdash \varphi$ where x does not occur in φ

Proof: $\Rightarrow$: If L has no theorems and $\{x\} \vdash_L \varphi$, then

$\emptyset \vdash_{L \amalg P_n} x \to_1 x \quad \{x \to_1 x\} \vdash_L \varphi$ imply $\emptyset \vdash_{L \amalg P_n} \varphi$

$\Leftarrow$: If $\Gamma \vdash_{L \amalg P_n} \varphi$ but $\Gamma \nvdash_L \varphi$, then rules of P_n must be used. $\emptyset \vdash \varphi \to \varphi$, then Modus ponens can be eliminated $\leadsto$ must be this + a rule as in 2.

If L has theorems, then could have used those instead.

Remote Algebraization

L is finitely algebraizable if $\exists$ binary formulas $\Delta = \{\Delta_1, \ldots, \Delta_k\}$ and unary formulas $\langle \delta, \varepsilon \rangle = \{\delta_1, \varepsilon_1, \ldots, \delta_\ell, \varepsilon_\ell\}$ s.t.

Cong.

eq.

(i) $\vdash_S \varphi \Delta \varphi$;

(ii) $\varphi \Delta \psi \vdash_S \psi \Delta \varphi$;

(iii) $\varphi \Delta \psi, \psi \Delta \vartheta \vdash_S \varphi \Delta \vartheta$;

For every n-ary connective ω

(iv) $\varphi_0 \Delta \psi_0, \ldots, \varphi_{n-1} \Delta \psi_{n-1} \vdash_S \omega \varphi_0 \ldots \varphi_{n-1} \Delta \omega \psi_0 \ldots \psi_{n-1}$.

(v) $\vartheta \dashv\vdash_S \delta(\vartheta) \Delta \epsilon(\vartheta)$.

L is (k,ℓ)-algebraizable if $|\Delta| = k$, $|\langle \delta, \varepsilon \rangle| = 2\ell$

Generic (k,ℓ)-algebraizable logic: $A_{k,\ell}$

[$\exists$ $A_{k,\ell} \longrightarrow L$ $\not\Rightarrow$ L (k,ℓ)-algebraizable]

Remote Algebraization

L remotely algebraizable via finite family

$\iff \exists$ conservative $L \longrightarrow L'$ ← (k,l)-algebraizable

$$\iff \quad \begin{tikzcd} L \arrow[r] \arrow[dr] & L \otimes A_{k,l} \arrow[d, dashed] & A_{k,l} \arrow[l] \\ & L' & \end{tikzcd}$$

Conservative

$L \otimes A_{k,l} := L \amalg A_{k,l}$ + extra rules for (iv)

Theorem: L is remotely algebraizable iff

1. L has theorems

or 2. L has no rule $\{x\} \vdash \varphi$ where x does not occur in φ

Proof: Messy.

Plan:

More on categories of logics

A diagram is filtered if $\forall$ $\begin{matrix} A_1 \\ \vdots \\ A_n \end{matrix}$

More on categories of logics

A diagram is filtered if $\forall \begin{matrix} A_1 \\ \vdots \\ A_n \end{matrix} \; \exists \; \begin{matrix} A_1 \to B \\ \vdots \\ A_n \to B \end{matrix}$

More on categories of logics

- A diagram is filtered if $\forall \; A_1, \dots, A_n \; \exists \; A_1 \to B, \dots, A_n \to B$

- An object B is finitely presentable if $\forall$ filtered diagrams D: $\operatorname{colim}_{d \in D} \operatorname{Hom}(B, d) \cong \operatorname{Hom}(B, \operatorname{colim} D)$

- A category is finitely accessible if it has filtered colims and every object is a filtered colim of fin. pres. objects.

Theorem (Makkai/Paré): An accessible category is the category of models of an infinitary first order theory.

A category is locally finitely presentable (l.f.p.) if it is finitely accessible and (co)complete.

Theorem (Gabriel/Ulmer/ ?):
An l.f.p. is a category of many-sorted partial algebras.

More on categories of logics

Theorem

- $\mathrm{Log}^{\mathrm{strict}}$ is an l.f.p. category.
- $\mathrm{Alg}^{\mathrm{strict}}$ is a finitely accessible category

A., Freire, Luciano, Mariano

- HoLog is an l.f.p. category

A., Mendes

Fin. pres. objects = logics over finite signatures generated by finitely many rules.

Plan:

Algebraizability as an algebraic structure

Have functors $\mathcal{A}_{k,\ell}\colon \mathrm{HoLog} \longrightarrow \mathrm{HoLog}$

$$L \longmapsto L \otimes A_{k,\ell}$$

translations: $L \longrightarrow L \otimes A_{k,\ell}$

$$L \otimes A_{k,\ell} \otimes A_{k,\ell} \longrightarrow L \otimes A_{k,\ell}$$

$\rightsquigarrow$ $\mathcal{A}_{k,\ell}$ is a (finitary) monad on HoLog

Algebraizability as an algebraic structure

Theorem: $\mathcal{A}_{k,l}$-Alg is the category of (k,l)-algebraizable logics and translations that preserve algebraizing pairs

Proof: L $\mathcal{A}_{k,l}$-algebra

$$\begin{tikzcd} L \arrow[r] \arrow[dr, equal] & L \otimes A_{k,l} \arrow[d] \\ & L \end{tikzcd}$$

has „dense" image
$\Rightarrow$ L (k,l)-algebraizable

$$\begin{tikzcd} L \otimes A_{k,l} \arrow[r, "f \otimes id"] \arrow[d] & L' \otimes A_{k,l} \arrow[d] \\ L \arrow[r, "f"'] & L' \end{tikzcd}$$

preserves algebraizing pair.

$\square$

Algebraizability as an algebraic structure

Corollary:

- $\mathcal{A}_{\kappa,\ell}$-Alg is l.f.p. [algebras over finitary monad on l.f.p. cat]
- HoAlg := cat. of algebraizable logics & alg. pair preserving translations

  is accessible

[HoAlg = $\operatorname{colim}_{\kappa,\ell} \mathcal{A}_{\kappa,\ell}$-Alg

Paré, Rosicky: directed colims of accessible cats are accessible]

Algebraizability as an algebraic structure

Remark: Codensity monad of HoAlg $\hookrightarrow$ HoLog:

$$\lim\left(\ldots \to L \otimes A_{k,k} \to L \otimes A_{k-1,k-1} \to \ldots \to L \otimes A_{1,1} \right)$$

– exists, but seems to tell little about a logic...

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Poset of all logics

Comparison with the poset of all logics (Jansana / Moraschini)

An interpretation $L \xrightarrow{f} L'$ is a (use finitary here!)
flex. translation s.t. $\mathrm{Mod}(L') \longrightarrow \mathrm{Mod}(L)$

$$(A, F) \longmapsto (A^f, F)$$

algebra reinterpreted along the signature morphism.

sends reduced matrices to reduced matrices.

$\rightsquigarrow$ category Int of logics and interpretations

Poset of all logics := poset reflection of Int

Poset of all logics

J./M.: The poset of all logics has
(now also infinitary)
set-indexed infima, but in general no finite suprema.

(Compare initial logic)

Poset reflection of HoLog is a complete lattice

JM: fin. pres. Leibniz class:

Sequence of fin. pres. equivalential logics $\{L_i\}_{i \in \mathbb{N}}$
$\rightsquigarrow$ $L \in$ corresponding Leibniz class
$\iff \exists$ interpretation $L_i \to L$ for some i

[Denseness not necessary]

This work suggests:

fin. pres. Leibniz classes correspond to accessible subcategories of HoLog.

Further Thoughts

Descent: Is L freely protoalgebraic?
I.e. $L \cong \tilde{L} \amalg P_1$ for some $\tilde{L}$?

If yes:

$$\begin{array}{ccc} L \amalg P_1 & & L \amalg P_1 \\ \cong & & \cong \\ \tilde{L} \amalg P_1 \amalg P_1 & \xrightarrow{\tau} & \tilde{L} \amalg P_1 \amalg P_1 \end{array}$$

[switch copies of P_1]

Then try to construct $\tilde{L}$ as equalizer of τ and id.

Obs.: This doesn't work with $L \otimes A_{k,l}$

– 2nd copy in $L \otimes A_{k,l} \otimes A_{k,l}$ satisfies more rules...

Further Thoughts

Thanks !