(A bit more) abstract Lindenbaum lemma

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TACL 2022

The result in one page ...

The "original" abstract Lindenbaum Lemma

Let $\vdash$ be a finitary consequence relation.
Then the meet-irreducible theories form a basis of $\mathrm{Th}(\vdash)$.

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Lindenbaum Lemma for certain infinitary consequence relations

Let $\vdash$ be a consequence relation

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Let $\vdash$ be a finitary consequence relation.
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Lindenbaum Lemma for certain infinitary consequence relations

Let $\vdash$ be a consequence relation on a countable set of formulas such that

- $\vdash$ has a countable axiomatization,
- $\mathrm{Th}(\vdash)$ is a frame,
- the intersection of any two finitely generated theories is finitely generated.

Then the finitely meet-irreducible theories form a basis of $\mathrm{Th}(\vdash)$.

An example of infinitary many-valued logic

The standard MV-algebra $[\mathbf{0},\mathbf{1}]_{\text{Ł}}$ has the real unit interval $[0,1]$ as domain and operations $\to$, $\&$, $\vee$, and $\neg$ interpreted as:

$$x \to y = \min\{1, 1 - x + y\} \qquad x \,\&\, y = \max\{0, x + y - 1\}$$

$$x \vee y = \max\{x, y\} \qquad \neg x = 1 - x$$

The logic of standard MV-algebra (a.k.a. infinitary Łukasiewicz logic):

$$\Gamma \models_{\mathrm{LSMVA}} \varphi \qquad \text{iff} \qquad (\forall e \colon \mathbf{Fm} \to [\mathbf{0},\mathbf{1}]_{\text{Ł}})(e[\Gamma] \subseteq \{1\} \Longrightarrow e(\varphi) = 1)$$

LSMVA is not finitary, e.g.:

$$\{\neg\varphi \to \varphi \,\&\, \overset{n}{\ldots} \,\&\, \varphi \mid n \geq 0\} \models_{\mathrm{LSMVA}} \varphi \qquad \text{but}$$

$$\{\neg\varphi \to \varphi \,\&\, \overset{n}{\ldots} \,\&\, \varphi \mid n \leq k\} \not\models_{\mathrm{LSMVA}} \varphi \qquad \text{for each } k$$

Two examples of infinitary modal logics

- In PDL:

$$\{[\alpha;\beta^n]\varphi \mid n \in \mathbb{N}\} \vDash [\alpha;\beta^*]\varphi$$

- In logics of common knowledge:

$$\{E^{n+1}\varphi \mid n \in \mathbb{N}\} \vDash C\varphi$$

A statement, a question, and some answers

To find strongly complete axiomatization of a given logic
we need (a variant of) Lindebaum lemma

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Can we have it for infinitary logics?

A very incomplete list of existing answers:

1963 Hay: (indirectly) for the Logic of Standard MV-Algebra

1977 Sundholm: for Von Wright's temporal logic

1993 Goldblatt: a general approach for modal logics with classical base

1994 Segerberg: a general method using saturated sets of formulas

2018 Bílková, Cintula, Lávička: a general method for certain algebraic logics

Consequence relations/logics

Fm: a countable set of formulas

A consequence relation $\vdash$ is a relation between sets of formulas and formulas s.t.:

- $\{\varphi\} \vdash \varphi$ (Reflexivity)
- If $\Gamma \vdash \varphi$, then $\Gamma \cup \Delta \vdash \varphi$ (Monotonicity)
- If $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for each $\psi \in \Gamma$, then $\Delta \vdash \varphi$ (Cut)

A consequence relation is

- finitary if: $\Gamma \vdash \varphi$ implies there is a finite $\Gamma' \subseteq \Gamma$ s.t. $\Gamma' \vdash \varphi$.
- structural (a.k.a. logic) if: $\Gamma \vdash \varphi$ implies $\sigma[\Gamma] \vdash \sigma(\varphi)$ for each substitution σ

CRs, COs, and CSs

Each CR determines a

- closure operator $\mathrm{Th}_{\vdash}() \colon \mathcal{P}(Fm) \to \mathcal{P}(Fm)$ defined as $\mathrm{Th}_{\vdash}(X) = \{a \mid X \vdash a\}$
- closure system $\mathrm{Th}(\vdash) \subseteq \mathcal{P}(Fm)$ defined as $\mathrm{Th}(\vdash) = \{X \mid X = \mathrm{Th}_{\vdash}(X)\}$

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An element of $\mathrm{Th}(\vdash)$ (a.k.a. a theory) is prime if it is not an intersection of two strictly bigger theories (i.e., it is finitely meet-irreducible)

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A set $\mathcal{X}$ of theories is a basis of $\mathrm{Th}(\vdash)$ if (any of the following),

- Whenever $X \nvdash a$, then there is a $T \in \mathcal{X}$ such that $X \subseteq T$ and $a \notin T$.
- Each theory is the intersection of a system of theories from $\mathcal{X}$.
- Each theory is the intersection of all theories from $\mathcal{X}$ extending it.

1st ingredient: Countable axiomatization

A CR $\vdash$ is countably axiomatizable if there is a countable set $\mathcal{AS} \subseteq \mathcal{P}(Fm) \times Fm$ st $X \vdash a$ iff there is a tree without infinite branches labeled by formulas st

- its root is labeled by a,
- if l is a label of some of its leafs, then $l \in X$ or $\emptyset \rhd l \in \mathcal{AS}$,
- if a non-leaf is labeled by c and P is the set of labels of its direct predecessors, then $P \rhd c \in \mathcal{AS}$.

1st ingredient: Countable axiomatization

Fact: each finitary CR (on countable set of formulas) is countably axiomatizable:

$$\mathcal{AS} = \{P \rhd c \mid P \vdash c \text{ and } P \text{ is finite}\}$$

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Not conversely: the logic $Ł_\infty$ given by Łukasiewicz 4 axioms, *MP*, and

$$\{\neg\varphi \to \varphi \,\&\, \overset{n}{\ldots} \,\&\, \varphi \mid n \geq 0\} \rhd \varphi \qquad \text{(Hay rule)}$$

is countably axiomatizable but not finitary as

$$\Gamma \vdash_{Ł_\infty} \varphi \qquad \text{implies} \qquad \Gamma \models_{\mathrm{LSMVA}} \varphi.$$

2nd ingredient: Nice lattice of theories

$\mathrm{Th}(\vdash)$ is domain of a complete lattice, where for $\mathcal{Y} \subseteq \mathrm{Th}(\vdash)$ we have:

$$\bigwedge \mathcal{Y} = \bigcap \mathcal{Y} \qquad\qquad \bigvee \mathcal{Y} = \mathrm{Th}_{\vdash}(\bigcup \mathcal{Y})$$

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$\mathrm{Th}(\vdash)$ is a frame if for each $\{X\} \cup \mathcal{Y} \subseteq \mathrm{Th}(\vdash)$:

$$X \cap \bigvee \mathcal{Y} = \bigvee_{Y \in \mathcal{Y}} (X \wedge Y).$$

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Fact: If $\vdash$ is finitary, then $\mathrm{Th}(\vdash)$ is frame iff it is distributive.

Fact (to be believed): $\mathrm{Th}(\vdash_{Ł_\infty})$ is a frame.

Recall that $\vdash_{Ł_\infty}$ is not finitary!

3rd ingredient: Disguised disjunctions

A CR has FGIP if the intersection of any two finitely generated theories is finitely generated

Fact: $\mathrm{Th}_{\vdash_{Ł_\infty}}(X) \cap \mathrm{Th}_{\vdash_{Ł_\infty}}(Y) = \mathrm{Th}_{\vdash_{Ł_\infty}}(\{x \vee y \mid x \in X, y \in Y\})$

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For a CR $\vdash$ with FGIP, then there is a function $\nabla\colon Fm \times Fm \to \mathcal{P}_{\text{fin}}(Fm)$ st.:

$$\mathrm{Th}_{\vdash}(x) \cap \mathrm{Th}_{\vdash}(y) = \mathrm{Th}_{\vdash}(x \nabla y).$$

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Fact: if $\mathrm{Th}(\vdash)$ is distributive, then

- existence of any such function ∇ implies FGIP
- (for any such function ∇) a theory T is prime iff

$$x \nabla y \subseteq T \quad \text{implies} \quad x \in T \text{ or } y \in T \qquad \text{(for each } x, y\text{)}$$

The main result

Lindenbaum Lemma for certain infinitary consequence relations

Let $\vdash$ be a consequence relation on a countable set of formulas such that

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Then the prime theories form a basis of $\mathrm{Th}(\vdash)$.

The proof — preparation: a relation $\Vdash \subseteq \mathcal{P}(Fm) \times \mathcal{P}_{\mathrm{fin}}(Fm)$

$$X \Vdash Y \qquad \text{iff} \qquad \bigcap_{y \in Y} \mathrm{Th}_{\vdash}(y) \subseteq \mathrm{Th}_{\vdash}(X).$$

If $\mathrm{Th}(\vdash)$ is a frame, then

$$\frac{\{X \Vdash Y \cup \{p\} \mid p \in P\} \qquad\qquad X \cup P \Vdash Y}{X \Vdash Y}.$$

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Set $W = \bigcap_{y \in Y} \mathrm{Th}_{\vdash}(y)$ we need to show that $W \subseteq \mathrm{Th}_{\vdash}(X)$

$$W \cap \mathrm{Th}_{\vdash}(P) = W \cap \bigvee_{p \in P} (\mathrm{Th}_{\vdash}(p)) = \bigvee_{p \in P} (W \cap \mathrm{Th}_{\vdash}(p)) \subseteq \mathrm{Th}_{\vdash}(X)$$

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$$\mathrm{Th}_{\vdash}(X) = (W \cap \mathrm{Th}_{\vdash}(P)) \vee \mathrm{Th}_{\vdash}(X) = (W \vee \mathrm{Th}_{\vdash}(X)) \cap (\mathrm{Th}_{\vdash}(P) \vee \mathrm{Th}_{\vdash}(X))$$
$$\supseteq (W \vee \mathrm{Th}_{\vdash}(X)) \cap W = W$$

The proof – assume that $X \nvdash x$

Enumerate rules of any $\mathcal{AS}$ of $\vdash$ as $P_i \rhd c_i$; assume that $\{y\} \rhd y \in \mathcal{AS}$ for each y

Construct sequences $X = X_0 \subseteq X_1 \subseteq \ldots$ and $\{x\} = Y_0 \subseteq Y_1 \subseteq \ldots$ st. $X_i \nVdash Y_i$:

$$\langle X_{i+1}, Y_{i+1} \rangle = \begin{cases} \langle X_i \cup \{c_i\}, Y_i \rangle & \text{if } X_i \cup \{c_i\} \nVdash Y_i \\ \langle X_i, Y_i \cup \{p\} \rangle & \text{for some } p \in P_i \text{ st. } X_i \nVdash Y_i \cup \{p\} \text{ otherwise} \end{cases}$$

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indeed such p has to exist (because $\mathrm{Th}(\vdash)$ if a frame):

$$\dfrac{\dfrac{\dfrac{\{X_i \Vdash Y_i \cup \{p\} \mid p \in P_i\}}{\{X_i \Vdash Y_i \cup \{c_i\} \cup \{p\} \mid p \in P_i\}} \qquad \dfrac{P_i \Vdash \{c_i\}}{X_i \cup P_i \Vdash Y_i \cup \{c_i\}}}{X_i \Vdash Y_i \cup \{c_i\}} \qquad X_i \cup \{c_i\} \Vdash Y_i}{X_i \Vdash Y_i}.$$

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Fact: $X' = \bigcup X_i$ is a theory.

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Fact: $X' = \bigcup X_i$ is a theory. Assume that $X' \vdash y$ and fix a proof of y from X'

We prove for each node that its label $l \in X_i$ for some i.

If it is a leaf and $l \in X'$, then it is trivial

and $\emptyset \rhd l$, then $l = c_i$ for some i and so $l \in X_i$

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We prove for each node that its label $l \in X_i$ for some i.

Otherwise there is a rule $P_i \rhd c_i$ st $l = c_i$ and each $p \in P$ is in X_j for some j (due to IH)

1st option $\Longrightarrow l = c_i \in X_{i+1}$

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1st option $\Longrightarrow l = c_i \in X_{i+1}$

2nd option $\Longrightarrow$ there is $p \in P_i$ and j st $p \in Y_{i+1} \cap X_j \subseteq Y_{\max\{i+1,j\}} \cap X_{\max\{i+1,j\}}$

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Fact: $X' = \bigcup X_i$ is a prime theory.

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Otherwise there are theories U_1 and U_2, elements $u_i \in U_i \setminus X$ and a finite set U st.

$$U \subseteq \mathrm{Th}_\vdash(U) = \mathrm{Th}_\vdash(u_1) \cap \mathrm{Th}_\vdash(u_2) \subseteq U_1 \cap U_2 = X$$

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Otherwise there are theories U_1 and U_2, elements $u_i \in U_i \setminus X'$ and a finite set U st.

$$U \subseteq \mathrm{Th}_{\vdash}(U) = \mathrm{Th}_{\vdash}(u_1) \cap \mathrm{Th}_{\vdash}(u_2) \subseteq U_1 \cap U_2 = X'$$

Thus there is i st. $U \subseteq X_i$ and $u_1, u_2 \in Y_i$ and so $X_i \Vdash Y_i$, a contradiction:

$$\bigcap_{y \in Y_i} \mathrm{Th}_{\vdash}(y) \subseteq \mathrm{Th}_{\vdash}(u_1) \cap \mathrm{Th}_{\vdash}(u_2) = \mathrm{Th}_{\vdash}(U) \subseteq \mathrm{Th}_{\vdash}(X_i)$$

The need for frames

Consider a CR $\vdash$ on the set $Fm = \mathbb{N}$ given by countable set of rules:

$$\{i \mid i > n\} \rhd n \qquad \text{(for } n \geq 0\text{)}$$

Fact 1: X is a theory iff $X = Fm$ or $Fm \setminus X$ is infinite

Fact 2: X is finitely generated iff X is finite; and so $\vdash$ has FGIP.

Fact 3: Fm is the only prime theory

Thus Lindenbaum lemma has to fail and $\mathrm{Th}(\vdash)$ is not a frame.

The need for countable axiomatization

Consider propositional language with $\vee$, and a constant $\mathbf{i}$ for each $i \in \mathbb{N}$.

Let $\vdash$ be the expansion of the disjunction-fragment of classical logic by:

$$\{\mathbf{i} \vee \chi \mid i \in C\} \rhd \chi$$

for each infinite set $C \subseteq \mathbb{N}$ and a formula χ.

Fact 1 (to believe): $\mathrm{Th}(\vdash)$ is a frame and $\vdash$ has FGIP

(as $\mathrm{Th}_{\vdash}(\chi) \cap \mathrm{Th}_{\vdash}(\psi) = \mathrm{Th}_{\vdash}(\chi \vee \psi)$)

Fact 2 (to believe): $X = \{2\mathbf{i} \vee 2\mathbf{i+1} \mid i \in \mathbb{N}\} \nvdash \mathbf{0}$

Fact 3: For each prime theory $T \supseteq \Gamma$ we have $T \vdash \mathbf{0}$; thus Lindenbaum lemma fails

The need for FGIP

?