The topological behaviour category of an algebraic theory

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Richard Garner
Centre of Australian Category Theory
Macquarie University

Algebraic theories in mathematics

A signature is a set Σ of operations σ, each with an arity $|\sigma| \in Set$

Example: signature Σ_{Grp} for groups is $\{\cdot, e, (-)^{-1}\}$ with arities $\{2,0,1\}$

Given a signature Σ and set A, can define set $\Sigma(A)$ of terms with free vars from A.

Example: we have $(x \cdot y) \cdot z$, $(x^{-1})^{-1}$, $(y \cdot e^{-1})^{-1} \cdot z$ $\in \Sigma_{Grp}(\{x,y,z\})$

An algebraic theory $\mathbb{T}$ is a signature Σ and a set E of equations $s = t$ between terms in the same free vars.

Example: $\mathbb{T}_{Grp}$ has equations like $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, $x \cdot x^{-1} = e$, ...

Algebraic theories in mathematics

If $\mathcal{C}$ is a category with products and $\mathbb{T}=(\Sigma,E)$ a theory, then a Σ-structure in $\mathcal{C}$ is an $X\in\mathcal{C}$ t/w interpretations

$$[\![\sigma]\!] : X^{|\sigma|} \longrightarrow X \qquad \text{for all } \sigma\in\Sigma.$$

Given a Σ-structure in $\mathcal{C}$, can recursively define derived interpretations

$$[\![t]\!] : X^{A} \longrightarrow X \qquad \text{for all } t\in\Sigma(A)$$

and say X is a $\mathbb{T}$-model if $[\![s]\!]=[\![t]\!]$ for all $s=t$ in E.

Example:
- A $\mathbb{T}_{Grp}$-model in $\mathbf{Set}$ is a group
- A $\mathbb{T}_{Grp}$-model in $\mathbf{Top}$ is a topological group
- A $\mathbb{T}_{Grp}$-model in $\mathbf{Cocoalg}_k$ is a cocommutative Hopf algebra

Algebraic theories in computer science

If $\mathbb{T}$ is an algebraic theory, and A is a set, then we write $T(A)$ for the set of $\mathbb{T}$-terms with free variables from A: this is the quotient of $\Sigma(A)$ by $\mathbb{T}$-provable equality.

Each $T(A)$ is a $\mathbb{T}$-model via substitution of terms; in fact, it's the free $\mathbb{T}$-model on the set A.

Example: $T_{\mathrm{Grp}}(A)$ is the free group with generating set A

In computer science, we see algebraic theories $\mathbb{T}$ as encoding notions of computation, with $T(A)$ being the set of all $\mathbb{T}$-programs returning values from the set A.

Algebraic theories in computer science

Example: Let I be a set. The theory $\mathbb{T}_{In}$ of input from the alphabet I has a single I-ary operation read, subject to no equations. We interpret elements of $T_{In}(A)$ as programs via

$$a \in A \subseteq T_{In}(A) \quad \leftrightsquigarrow \quad \text{return } a$$

$$\mathrm{read}(\lambda i.\ t_i) \quad \leftrightsquigarrow \quad \text{let } i := \mathrm{read}() \text{ in } t(i)$$

For instance, when $I = \mathbb{N}$, the program which reads two numbers and returns their sum is given by

$$\mathrm{read}(\lambda n.\ \mathrm{read}(\lambda m.\ n+m)) \in T_{In}(\mathbb{N})$$

Algebraic theories in computer science

Example: The theory $\mathbb{T}_{RI}$ of reversible input from the alphabet I extends $\mathbb{T}_{In}$ with I new unary operations unread_i and equations

$$\mathrm{unread}_i(\mathrm{read}(\lambda j.\ x_j)) = x_i \qquad \forall i \in I$$
$$\mathrm{read}(\lambda i.\ \mathrm{unread}_i(x)) = x$$

We think of $\mathrm{unread}_i(x)$ as pushing i back onto the input stream and then continuing as x.

For instance, when $I = \mathbb{N}$, the program which reads two numbers and puts their sum back on the input stream is given by

$$\mathrm{read}(\lambda n.\ \mathrm{read}(\lambda m.\ \mathrm{unread}_{n+m}(*))) \in T_{RI}(\{*\}).$$

Algebraic theories in computer science

Definition: A *self-similar action* of a monoid M on I is a function

$$\delta : I \times M \longrightarrow M \times I \qquad (i, m) \longmapsto (m|_i \, , i \cdot m)$$

such that

$$i \cdot 1 = i \qquad\qquad 1|_i = 1$$

$$(i \cdot m) \cdot n = i \cdot (mn) \qquad\qquad mn|_i = m|_i \, n|_{i \cdot m}$$

Idea: δ induces an **action** of M on $I^{\mathbb{N}}$ via

$$(\ldots, i_2, i_1, i_0) \cdot m = (\ldots, i_2 \cdot (m|_{i_0})|_{i_1}, i_1 \cdot m|_{i_0}, i_0 \cdot m)$$

Example: let $M = \mathbb{N}$ and $I = \{0, 1\}$. The **adder** action is:

$$(i, 2n) \longmapsto (n, i)$$

$$(0, 2n+1) \longmapsto (n, 1)$$

$$(1, 2n+1) \longmapsto (n+1, 0)$$

and, e.g., $(\cdots 0 \; 1 \; 1 \; 0) \cdot 3 = (\cdots 1 \; 0 \; 0 \; 1)$

Algebraic theories in computer science

Example: Let $\delta : I \times M \longrightarrow M \times I$ be a self-similar monoid action. The theory $\mathbb{T}_\delta$ of reversible input acted on by M via δ extends $\mathbb{T}_{RI}$ with unary operations $(\alpha_m : m \in M)$ and equations

$$\alpha_1(x) = x \qquad \alpha_m(\alpha_n(x)) = \alpha_{mn}(x)$$

$$\alpha_m(\mathrm{read}(\lambda i.\, x_i)) = \mathrm{read}(\lambda i.\, \alpha_{m|_i}(x_{m \cdot i}))$$

The idea is that $\alpha_m(x)$ acts on the input stream via $(-)\cdot m$ and then continues as x. For instance, when δ is as on last slide, the program which adds the font four bits of the stream to the rest is

$$\mathrm{read}(\lambda i_0.\, \mathrm{read}(\lambda i_1.\, \mathrm{read}(\lambda i_2.\, \mathrm{read}(\lambda i_3.\, \alpha_{i_0+2i_1+4i_2+8i_3}(*))) \in \mathbb{T}_\delta(\{*\})$$

Algebraic theories in computer science

Example: Let B be a Boolean algebra. The theory $\mathbb{T}_B$ of B-valued Boolean state has binary operations b for each $b \in B$ and equations

Bergman
1991

$$b(x,x) = x \qquad b(b(x,y),z) = b(x,z) \qquad b(x, b(y,z)) = b(x,z)$$

$$1(x,y) = x \qquad b'(x,y) = b(y,x) \qquad b(c(x,y),y) = (b \wedge c)(x,y)$$

The idea is that B is a Boolean algebra of propositions about the external world, and that

$$b(x,y) \quad \leftrightsquigarrow \quad \text{if } b \text{ then } x \text{ else } y$$

Comodels

In mathematics, we care about algebraic theories for their models.

In computer science, we care more about free models ... but also comodels!

A comodel of an algebraic theory $\mathbb{T}$ is a model in $\mathbf{Set}^{op}$. Thus, it involves:

- A set S
- For each $\sigma \in \Sigma$ a co-interpretation $[\![\sigma]\!] : S \to |\sigma| \times S$;
- ... inducing derived co-interpretations $[\![t]\!] : S \to A \times S$ for all $t \in \Sigma(A)$;
- ... which we require to satisfy $[\![t]\!] = [\![u]\!]$ for all $t = u$ in E

In mathematics, comodels tend to be rather dull:

Example: A $\mathbb{T}_{Grp}$-comodel S involves, among other things, a cointerpretation $[\![e]\!] : S \longrightarrow \phi$, which forces $S = \phi$.

Comodels

… but in computer science, comodels are much more interesting!

Example: A $\mathbb{T}_{\text{In}}$-comodel is a set S t/w a function $[\![\text{read}]\!] : S \to I \times S$.
We view this as a state machine that answers requests for I-tokens:

- S is a set of states;
- $[\![\text{read}]\!]$ assigns to each state $s \in S$ a next token $i \in I$ and a next state $s' \in S$.

Power-Shkaravska 2004

In general, if $\mathbb{T}$-terms are programs which interact with an environment, then $\mathbb{T}$-comodels are state machines providing instances of that environment.

Comodels

In this view, the cointerpretation $[\![t]\!] : S \to A \times S$ of $t \in T(A)$ assigns to each $s \in S$ the result of running the computation t from state s to get a return value $a \in A$ and a final state $s' \in S$.

Example: let $S = \{a, b, c\}$ be the $\mathbb{T}_{\mathrm{In}}$-comodel over the alphabet $\mathbb{N}$ with:

$$[\![\mathrm{read}]\!] : \quad a \longmapsto (7, b) \qquad b \longmapsto (11, c) \qquad c \longmapsto (9, b)$$

The co-interpretation $[\![t]\!] : S \to \mathbb{N} \times S$ of $t = \mathrm{read}(\lambda n.\, \mathrm{read}(\lambda m.\, n+m)) \in T_{\mathrm{In}}(\mathbb{N})$ is:

$$[\![t]\!] : \quad a \longmapsto (18, c) \qquad b \longmapsto (20, b) \qquad c \longmapsto (20, c)$$

Comodels

Example: A $\mathbb{T}_{RI}$-comodel is a set S t/w functions

$$[\![\mathrm{read}]\!] : S \to I \times S \qquad \left([\![\mathrm{unread}_i]\!] : S \to S\right)_{i \in I}$$

... and the axioms say $[\![\mathrm{read}]\!]$ is inverse to $\langle [\![\mathrm{unread}_i]\!] \rangle_{i \in I} : I \times S \to S$.

So a comodel is a set S with an isomorphism $S \cong I \times S$.

Example: If $\delta : I \times M \to M \times I$ is a self-similar monoid action, then a $\mathbb{T}_\delta$-comodel is a $\mathbb{T}_{RI}$-comodel S with a right M-action s/t:

$$[\![\mathrm{read}]\!](s) = (i, s') \implies [\![\mathrm{read}]\!](s \cdot m) = (i \cdot m, s' \cdot m|_i)$$

Example: If $\mathbb{B}$ is a Boolean algebra, then a $\mathbb{T}_B$-comodel is a set S t/w a function $S \times B \xrightarrow{v} 2$ s.t. each $v(s,-) : B \to 2$ is a Boolean homomorphism.

The final comodel

Given a $\mathbb{T}_{\mathrm{In}}$-comodel

$$[\![\mathrm{read}]\!] : S \longrightarrow I \times S \qquad s \longmapsto (h(s), \partial(s))$$

each state $s \in S$ has an associated behaviour: the stream of values

$$\beta(s) := \big(h(s), h(\partial(s)), h(\partial^2(s)), h(\partial^3(s)), \dots \big)$$

Abstractly, we find these behaviours as elements of the final comodel

$$[\![\mathrm{read}]\!] : I^{\mathbb{N}} \longrightarrow I \times I^{\mathbb{N}} \qquad \underbrace{(i_0, i_1, i_2, \dots)}_{i} \longmapsto (i_0, \underbrace{(i_1, i_2, \dots)}_{\partial(i)})$$

and recover $\beta(s)$ from s via the unique comodel map

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\beta"'] & I \times S \arrow[d, "I \times \beta"] \\ I^{\mathbb{N}} \arrow[r] & I \times I^{\mathbb{N}} \end{tikzcd}$$

In fact we can describe the final comodel for a general $\mathbb{T}$!

The final comodel

Definition: Let $\mathbb{T}$ be an algebraic theory, $t \in T(I)$ and $u \in T(J)$.
We write

$$t \gg u := t(\lambda i.\ u) \in T(J)$$

(run t, throw away the return value, and then run u).

Definition: Let $\mathbb{T}$ be an algebraic theory. An admissible $\mathbb{T}$-behaviour is is a natural family of functions

$$\beta_I : T(I) \longrightarrow I$$

such that, for all $t \in T(I)$ and $\vec{u} \in T(J)^I$, we have

$$\beta(t(\vec{u})) = \beta(t \gg u_{\beta(t)})$$

The final comodel

Theorem (G.): Let $\mathbb{T}$ be an algebraic theory. The final $\mathbb{T}$-comodel F is the set of $\mathbb{T}$-admissible behaviours, with cooperations

$$\begin{aligned} [\![\sigma]\!] : F &\longrightarrow |\sigma| \times F \\ \beta &\longmapsto (\beta(\sigma), \beta(\sigma \gg -)) \end{aligned}$$

Example: an admissible behaviour of $\mathbb{T}_{\mathrm{In}}$ is uniquely specified by

$$\Big(\underset{\substack{\shortparallel \\ i_0}}{\beta(\mathrm{read})}, \ \beta(\underset{\substack{\shortparallel \\ i_1}}{\mathrm{read} \gg \mathrm{read}}), \ \beta(\underset{\substack{\shortparallel \\ i_2}}{\mathrm{read} \gg \mathrm{read} \gg \mathrm{read}}), \ \dots \Big) \quad \in I^{\mathbb{N}}$$

E.g., $\beta(\mathrm{read}(\lambda n.\ \mathrm{read}(\lambda m.\ n+m))) = \beta(\mathrm{read} \gg \mathrm{read}(\lambda m.\ i_0 + m))$

$= \beta(\mathrm{read} \gg \mathrm{read} \gg i_0 + i_1) \ = i_0 + i_1$

The final comodel

Example: The final $\mathbb{T}_{RI}$-comodel is once again $I^{\mathbb{N}}$, with $[\![\mathrm{read}]\!]$ given as before, and with

$$[\![\mathrm{unread}_i]\!] : I^{\mathbb{N}} \longrightarrow I^{\mathbb{N}} \qquad (i_0, i_1, \dots) \longmapsto (i, i_0, i_1, \dots)$$

Example: If $\delta : I \times M \to M \times I$ is a self-similar monoid action, then the final $\mathbb{T}_\delta$-comodel is the final $\mathbb{T}_{RI}$-comodel augmented by the right M-action:

$$(i_0, i_1, i_2, \dots) \cdot m = (i_0 \cdot m,\ i_1 \cdot (m|_{i_0}),\ i_2 \cdot (m|_{i_0}|_{i_1}), \dots)$$

Example: if $\mathbb{B}$ is a Boolean algebra, the final $\mathbb{T}_{\mathbb{B}}$-comodel is $\mathcal{U}\mathbb{B} = \mathrm{BAlg}(\mathbb{B}, 2)$ with

$$v : \mathcal{U}\mathbb{B} \times \mathbb{B} \longrightarrow 2 \qquad v(\varphi, b) = \varphi(b)$$

The behaviour category

So we now understand the final comodel pretty well. What about an arbitrary comodel?

Theorem (G.): Let $\mathbb{T}$ be an algebraic theory. The category of $\mathbb{T}$-comodels is a presheaf category $[\mathbb{B}_{\mathbb{T}}, \mathrm{Set}]$, where the behaviour category $\mathbb{B}_{\mathbb{T}}$ has:

- objects being admissible behaviours (ie $\mathrm{ob}\,\mathbb{B}_{\mathbb{T}} = F$);
- $\mathbb{B}_{\mathbb{T}}(\beta, \gamma) = \{m \in T(1) : \gamma = \beta(m(-))\} / \sim_\beta$

  where $\sim_\beta$ is smallest equiv. relation s.t.

  $$t(\lambda i.\, m_i) \sim_\beta t \gg m_{\beta(t)}$$

The behaviour category

Example: The behaviour category $\mathbb{B}_{\mathrm{In}}$ of Π_{In} has:

- object-set $I^{\mathbb{N}}$;
- $\mathbb{B}(\vec{i},\vec{j}) = \{ n \in \mathbb{N} : \partial^n \vec{i} = \vec{j} \}$

E.g. $\cdots 0101110 \xrightarrow{3} \cdots 0101$

Example: The behaviour category $\mathbb{B}_{\mathrm{RI}}$ of Π_{RI} has:

- object-set $I^{\mathbb{N}}$;
- $\mathbb{B}(\vec{i},\vec{j}) = \{ k \in \mathbb{Z} : \partial^{N+k} \vec{i} = \partial^{N} \vec{j} \text{ for some } N \in \mathbb{N} \}$

E.g.

$$\begin{tikzcd} \cdots 0101110 \arrow[r, shift left, "3"] & \cdots 0110 \arrow[l, shift left, "-3"] \end{tikzcd}$$

The behaviour category

Example: for a self-similar action $I \times M \xrightarrow{\delta} M \times I$, the behaviour category $\mathbb{B}_{\delta}$ has:

- object-set $I^{\mathbb{N}}$;
- $\mathbb{B}(\vec{\imath}, \vec{\jmath}) = \{ (r, m, s) \in \mathbb{N} \times M \times \mathbb{N} : \partial^{s}(\vec{\jmath}) = \partial^{r}(\vec{\imath}) \cdot m \} / \sim$

where $\sim$ generated by $(r, m, s) \sim (r+1, m|_{i_r}, s+1)$.

E.g. $\overbrace{\cdots 100101110}^{w} \xrightarrow{(5, m, 2)} \overbrace{\cdots 111110}^{w \cdot m}$

Example: for a Boolean algebra B, the behaviour category $\mathbb{B}_B$ is discrete on $\mathcal{U}B = \mathrm{BAlg}(B, 2)$.

Topological comodels

A topological comodel of an algebraic theory $\mathbb{T}$ is a model in Top^{op}.

Thus, it's a comodel S t/w a topology on S making each $[\![\sigma]\!] : S \to |\sigma| \times S$ continuous.

discrete topology

Idea: open sets in S encode computably observable sets of states.

Theorem (G.): Let $\mathbb{T}$ be an algebraic theory. The final topological comodel is the final $\mathbb{T}$-comodel F under the topology w/ subbasis:

$$[t \mapsto i] = \{\beta \in F : \beta(t) = i\} \qquad \forall\, t \in T(I),\ i \in I.$$

Topological comodels

Example: In the cases of $\mathbb{T}_{\mathrm{In}}$, $\mathbb{T}_{RI}$ and $\mathbb{T}_{S}$, the topology on the final topological comodel $I^{\mathbb{N}}$ is the prodiscrete (= Baire) topology, with basic clopens

$$[i_0 \cdots i_n] = \{ \vec{i} \in I^{\mathbb{N}} : \vec{i}|_{0,\dots,n} = (i_0, \dots, i_n) \}$$

Example: In the case of $\mathbb{T}_B$ for a Boolean algebra B, the topology on the final topological comodel $\mathcal{U}B$ is the Stone topology, with basic clopens

$$[b] = \{ \varphi : B \to 2 \mid \varphi(b) = \top \}$$

The topological behaviour category

So we now understand the final topological comodel. What about an arbitrary topological comodel?

Theorem (G.): Let $\mathbb{T}$ be an algebraic theory. The category of topological comodels is the category of left $\mathbb{B}_{\mathbb{T}}$-spaces, where $\mathbb{B}_{\mathbb{T}}$ is the topological behaviour category with

- Object-space the final topological comodel F;
- Arrow-space the topologisation of the arrows of the behaviour caty w/ subbasic open sets

$$[m, t \mapsto i] = \{ m : \beta \longrightarrow \beta(m(-)) \mid \beta(t) = i \}$$ for $m \in T(1), t \in T(I), i \in I$

The topological behaviour category

In our examples, the topological behaviour categories give known objects from the world of non-commutative geometry:

- In the case of Π_{RI}, we get the Cuntz topological groupoid, whose associated C^*-algebra is the Cuntz C^*-algebra and whose associated $\mathbb{R}$-algebra is the Leavitt algebra.
- In the case of Π_δ for $G \times I \xrightarrow{\delta} I \times G$ a self-similar group action, we get the Nekrashevych - Röver groupoid.
- ... just the start of a bigger story!

The bigger picture

An obvious question: which topological catys are behaviour catys?

We can in fact over-answer this question. There's an adjunction

finitary for simplicity

$$\begin{tikzcd} \mathrm{AlgThy}^{\omega} \arrow[r, shift left=2, "\Pi \mapsto \mathbb{B}_{\Pi}"] \arrow[r, phantom, "\perp"] & (\mathrm{TopCat})^{op} \arrow[l, shift left=2, "\Pi_{\mathbb{C}} \leftarrow\!\shortmid \mathbb{C}"] \end{tikzcd}$$

morphisms are cofunctors

where $\Pi_{\mathbb{C}}$ extends $\Pi_{\mathrm{clopen}(C_0)}$ with unary ops m for each $\begin{tikzcd} C_1 \arrow[d, "s"] \\ C_0 \arrow[u, bend left, dotted, "m"] \end{tikzcd}$ + axioms.

This is a Galois (= idempotent) adjunction whose restriction to fixpoints is:

$$\begin{tikzcd} \mathrm{CartClosedThy}^{\omega} \arrow[r, bend left] \arrow[r, phantom, "\simeq"] & (\mathrm{AmpleTopCat})^{op} \arrow[l, bend left] \end{tikzcd}$$

induces cart closed variety

source map is étale
space of obs is Stone space

This extends the non-commutative Stone duality of Kudryavtseva & Lawson