Uniform Lyndon Interpolation for Basic Non-normal Modal and Conditional Logics

Raheleh Jalali

(joint work with Amir Akbar Tabatabai and Rosalie Iemhoff)

Utrecht University

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Non-normal Modal Logics

Languages:

(modal) $\mathcal{L}_\Box = \{\bot, \wedge, \vee, \rightarrow, \Box\}$, (conditional) $\mathcal{L}_\rhd = \{\bot, \wedge, \vee, \rightarrow, \rhd\}$

$\mathcal{L}$ refers to both $\mathcal{L}_\Box$ and $\mathcal{L}_\rhd$. Define $\top := \bot \rightarrow \bot$, $\neg A := A \rightarrow \bot$, and $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$.

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- Logic E: the smallest set of formulas in $\mathcal{L}_\Box$ containing all classical tautologies and closed under:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\ MP \qquad\qquad \frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}\ E$$

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Modal axioms: M and C are weakenings of $\Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B$

- $M : \Box(A \wedge B) \to \Box A \wedge \Box B$
- $C : \Box A \wedge \Box B \to \Box(A \wedge B)$
- $N : \Box\top$

Other Non-normal Modal Logics

Non-normal modal logics are defined by adding the modal axioms to the base logic E:

$$\mathsf{EN} = \mathsf{E} + (N) \qquad \mathsf{M} = \mathsf{E} + (M)$$

$$\mathsf{MN} = \mathsf{M} + (N) \qquad \mathsf{MC} = \mathsf{M} + (C)$$

$$\mathsf{K} = \mathsf{MC} + (N) \qquad \mathsf{EC} = \mathsf{E} + (C)$$

$$\mathsf{ECN} = \mathsf{EC} + (N)$$

Conditional Logics

- Logic CE: the smallest set of formulas in $\mathcal{L}_{\rhd}$ containing all classical tautologies and closed under:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\, MP \qquad\qquad \frac{\varphi_0 \leftrightarrow \varphi_1 \qquad \psi_0 \leftrightarrow \psi_1}{\varphi_0 \rhd \psi_0 \to \varphi_1 \rhd \psi_1}\, CE$$

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Conditional axioms: weakening of $(\varphi \rhd (\psi \wedge \theta)) \leftrightarrow (\varphi \rhd \psi) \wedge (\varphi \rhd \theta)$

- *CM* : $(\varphi \rhd (\psi \wedge \theta)) \to (\varphi \rhd \psi) \wedge (\varphi \rhd \theta)$
- *CC* : $(\varphi \rhd \psi) \wedge (\varphi \rhd \theta) \to (\varphi \rhd (\psi \wedge \theta))$
- *CN* : $\varphi \rhd \top$
- *CEM* : $(\varphi \rhd \psi) \vee (\varphi \rhd \neg\psi)$
- *ID* : $\varphi \rhd \varphi$

Other Conditional Logics

Conditional logics are defined by adding the conditional axioms to CE:

$$\begin{aligned}
\mathrm{CEN} &= \mathrm{CE} + (CN) & \mathrm{CM} &= \mathrm{CE} + (CM) \\
\mathrm{CMN} &= \mathrm{CM} + (CN) & \mathrm{CMC} &= \mathrm{CM} + (CC) \\
\mathrm{CK} &= \mathrm{CMC} + (CN) & \mathrm{CEC} &= \mathrm{CE} + (CC) \\
\mathrm{CECN} &= \mathrm{CEC} + (CN) & \mathrm{CKID} &= \mathrm{CK} + (ID) \\
\mathrm{CKCEM} &= \mathrm{CK} + (CEM) & \mathrm{CKCEMID} &= \mathrm{CKCEM} + (ID)
\end{aligned}$$

The Polarity of the Variables

Positive and Negative Variables

- $V^+(p)=\{p\}$, $V^-(p)=V^+(\top)=V^-(\top)=V^+(\bot)=V^-(\bot)=\varnothing$, for atom p,
- $V^+(\varphi \odot \psi) = V^+(\varphi) \cup V^+(\psi)$ and $V^-(\varphi \odot \psi) = V^-(\varphi) \cup V^-(\psi)$, for $\odot \in \{\wedge, \vee\}$,
- $V^+(\varphi \to \psi) = V^-(\varphi) \cup V^+(\psi)$ and $V^-(\varphi \to \psi) = V^+(\varphi) \cup V^-(\psi)$,
- $V^+(\Box\varphi) = V^+(\varphi)$ and $V^-(\Box\varphi) = V^-(\varphi)$, for $\mathcal{L} = \mathcal{L}_\Box$.
- $V^+(\varphi \rhd \psi) = V^-(\varphi) \cup V^+(\psi)$ and $V^-(\varphi \rhd \psi) = V^+(\varphi) \cup V^-(\psi)$, for $\mathcal{L} = \mathcal{L}_\rhd$.

For an atomic formula p, a formula φ is called p^+*-free* (p^-*-free*), if $p \notin V^+(\varphi)$ ($p \notin V^-(\varphi)$). For a sequent $S = (\Gamma \Rightarrow \Delta)$, define $V^+(S)$ (res. $V^-(S)$) as $V^+(\bigwedge \Gamma \to \bigvee \Delta)$ (res. $V^-(\bigwedge \Gamma \to \bigvee \Delta)$).
$V(\varphi) := V^+(\varphi) \cup V^-(\varphi)$.

We use $\circ, \dagger \in \{+, -\}$ as variables for $+$ and $-$ and $\diamond$ for the dual of $\circ$.

Lyndon Interpolation

Lyndon Interpolation Property (LIP)

A logic L has *Lyndon interpolation property* (LIP) if for any formulas $\varphi, \psi \in \mathcal{L}$ such that $L \vdash \varphi \to \psi$, there is a formula $\theta \in \mathcal{L}$ such that

1. $V^+(\theta) \subseteq V^+(\varphi) \cap V^+(\psi)$;
2. $V^-(\theta) \subseteq V^-(\varphi) \cap V^-(\psi)$;
3. $L \vdash \varphi \to \theta$;
4. $L \vdash \theta \to \psi$.

A logic has *Craig interpolation property* (CIP) if it has the above properties, omitting all the $+$ and $-$ superscripts.

Uniform Lyndon Interpolation Property (ULIP)

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A logic L has *ULIP* if for any formula $\varphi \in \mathcal{L}$, atom p, and $\circ \in \{+, -\}$, there are p°-free formulas, $\forall^\circ p\varphi$ and $\exists^\circ p\varphi$, such that $V^\dagger(\exists^\circ p\varphi) \subseteq V^\dagger(\varphi)$ and $V^\dagger(\forall^\circ p\varphi) \subseteq V^\dagger(\varphi)$, for any $\dagger \in \{+, -\}$ and

- $L \vdash \forall^\circ p\varphi \to \varphi$,
- for any p°-free formula ψ if $L \vdash \psi \to \varphi$ then $L \vdash \psi \to \forall^\circ p\varphi$,
- $L \vdash \varphi \to \exists^\circ p\varphi$, and
- for any p°-free formula ψ if $L \vdash \varphi \to \psi$ then $L \vdash \exists^\circ p\varphi \to \psi$.

A logic has *uniform interpolation property (UIP)* if it has all the above properties, omitting the superscripts $\circ, \dagger \in \{+, -\}$, everywhere.

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- $L \vdash \forall^\circ p\varphi \to \varphi$,
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- for any p°-free formula ψ if $L \vdash \varphi \to \psi$ then $L \vdash \exists^\circ p\varphi \to \psi$.

A logic has *uniform interpolation property (UIP)* if it has all the above properties, omitting the superscripts $\circ, \dagger \in \{+, -\}$, everywhere.

Theorem

If a logic L has ULIP, then it has both LIP and UIP.

Main Results

Theorem (Akbar Tabatabai, Iemhoff, J.)

The following logics have ULIP (and hence UIP and LIP):

modal	E, M, EN, MN, MC, K
conditional	CE, CM, CEN, CMN, CMC, CK, CKID

The following logics do not have ULIP (but they have UIP):

- CKCEM *and* CKCEMID

The following logics do not have CIP (and hence no U(L)IP):

modal	EC, ECN
conditional	CEC, CECN

The systems **G3cp** and **G3w**

Consider the following system for classical logic called **G3cp**:

$$\frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \frac{}{\Gamma, \bot \Rightarrow, \Delta}$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\, L\wedge \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}\, R\wedge$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\, R\vee$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \rightarrow \psi \Rightarrow \Delta}\, L\rightarrow \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \rightarrow \psi, \Delta}\, R\rightarrow$$

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$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \rightarrow \psi \Rightarrow \Delta}\ L\rightarrow \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \rightarrow \psi, \Delta}\ R\rightarrow$$

Moreover, define **G3w** as **G3cp** plus the following weakening rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ Lw \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ Rw$$

Sequent Calculi for Basic Non-normal Modal Logics

Consider the following modal rules to add to **G3w** to produce a cut-free system for their corresponding logics [Orlandelli 2019]:

$$\frac{\varphi \Rightarrow \psi \qquad \psi \Rightarrow \varphi}{\Box\varphi \Rightarrow \Box\psi}\, E \qquad \frac{\Rightarrow \psi}{\Rightarrow \Box\psi}\, N$$

$$\frac{\varphi \Rightarrow \psi}{\Box\varphi \Rightarrow \Box\psi}\, M \qquad \frac{\varphi_1, \cdots, \varphi_n \Rightarrow \psi}{\Box\varphi_1, \cdots, \Box\varphi_n \Rightarrow \Box\psi}\, MC$$

Note that in each rule (both modal and propositional), the weight of each of the premises (sum of the length of its formulas) is less than the weight of the consequence and hence the systems are terminating.

GM = **G3w** + M

G3w plus the following rules:

$$\frac{\varphi_0 \Rightarrow \varphi_1 \qquad \varphi_1 \Rightarrow \varphi_0 \qquad \psi_0 \Rightarrow \psi_1 \qquad \psi_1 \Rightarrow \psi_0}{\varphi_1 \rhd \psi_1 \Rightarrow \varphi_0 \rhd \psi_0}\ CE$$

$$\frac{\varphi_0 \Rightarrow \varphi_1 \qquad \varphi_1 \Rightarrow \varphi_0 \qquad \psi_1 \Rightarrow \psi_0}{\varphi_1 \rhd \psi_1 \Rightarrow \varphi_0 \rhd \psi_0}\ CM$$

$$\frac{\{\varphi_0 \Rightarrow \varphi_i \quad , \quad \varphi_i \Rightarrow \varphi_0\}_{1 \leqslant i \leqslant n} \qquad \psi_1, \cdots, \psi_n \Rightarrow \psi_0}{\varphi_1 \rhd \psi_1, \cdots, \varphi_n \rhd \psi_n \Rightarrow \varphi_0 \rhd \psi_0}\ CMC\ (n \geqslant 1)$$

$$\frac{\Rightarrow \psi_0}{\Rightarrow \varphi_0 \rhd \psi_0}\ CN$$

$$\frac{\{\varphi_0 \Rightarrow \varphi_i \quad , \quad \varphi_i \Rightarrow \varphi_0\}_{i\in I} \qquad \varphi_0, \{\psi_i\}_{i\in I} \Rightarrow \psi_0}{\{\varphi_i \rhd \psi_i\}_{i\in I} \Rightarrow \varphi_0 \rhd \psi_0} \; CKID$$

$$\frac{\{\varphi_0 \Rightarrow \varphi_r \quad , \quad \varphi_r \Rightarrow \varphi_0\}_{r\in I\cup J} \qquad \{\psi_i\}_{i\in I} \Rightarrow \psi_0, \{\psi_j\}_{j\in J}}{\{\varphi_i \rhd \psi_i\}_{i\in I} \Rightarrow \varphi_0 \rhd \psi_0, \{\varphi_j \rhd \psi_j\}_{j\in J}} \; CKCEM$$

$$\frac{\{\varphi_0 \Rightarrow \varphi_r \quad , \quad \varphi_r \Rightarrow \varphi_0\}_{r\in I\cup J} \qquad \varphi_0, \{\psi_i\}_{i\in I} \Rightarrow \psi_0, \{\psi_j\}_{j\in J}}{\{\varphi_i \rhd \psi_i\}_{i\in I} \Rightarrow \varphi_0 \rhd \psi_0, \{\varphi_j \rhd \psi_j\}_{j\in J}} \; CKCEMID$$

- Cut-free terminating systems;

ULIP for Sequents

We extend interpolation from logics to sequent calculi. As we are in classical setting, we only define $\forall^{\circ} p S$ and $\exists^{\circ} p S$ is $\neg \forall^{\diamond} p \neg S$.

ULIP for sequent calculi

G has ULIP if for any sequent $S = (\Gamma \Rightarrow \Delta)$, any atom p and any $\circ \in \{+, -\}$, there exists a formula $\forall^{\circ} p S$ such that:

(var) $\forall^{\circ} p S$ is p°-free and $V^{\dagger}(\forall^{\circ} p S) \subseteq V^{\dagger}(S)$, for any $\dagger \in \{+, -\}$,

(i) $\Gamma, \forall^{\circ} p S \Rightarrow \Delta$ is derivable in G,

(ii) for any sequent $\Sigma \Rightarrow \Lambda$ such that $p \notin V^{\diamond}(\Sigma \Rightarrow \Lambda)$, if $\Gamma, \Sigma \Rightarrow \Delta, \Lambda$ is derivable in G then $(\Sigma \Rightarrow \forall^{\circ} p S, \Lambda)$ is derivable in G.

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Theorem

Let G be one of the sequent calculi introduced here and L be its logic. Then, G has ULIP (resp., UIP) iff L has ULIP (resp., UIP).

Proof Idea

Let G be one of the sequent calculi introduced.

- Backward application of any of the rules in G decreases the weight of the sequent.
- Using this property and *recursion on the weight of the sequents*, for any given sequent S, any atom p and any $\circ \in \{+, -\}$, we define a p°-free formula $\forall^{\circ} p S$.
- Then by *induction on the weight* of S, we prove that $\forall^{\circ} p S$ meets all the required conditions in Definition in the previous slide.

Proof Sketch for M (using **GM**)

Define a formula $\forall^{\circ} pS$ by recursion on the weight of $S = (\Gamma \Rightarrow \Delta)$: if S is provable define $\forall^{\circ} pS$ as $\top$, otherwise, define it as:

$$(\forall^{\circ}_{ax} pS) \vee \bigvee_{R} (\bigwedge_{i} \forall^{\circ} pS_i) \vee (\forall^{\circ}_{m} pS)$$

$\forall^{\circ}_{ax} pS$ is the disjunction of all p°-free formulas in Δ and the negation of all $p^{\diamond}$-free formulas in Γ (where $S = (\Gamma \Rightarrow \Delta)$).

Proof Sketch for M

$$(\forall^\circ_{ax} pS) \vee \bigvee_R (\bigwedge_i \forall^\circ pS_i) \vee (\forall^\circ_m pS)$$

the second disjunction is over all rules R in **G3w** backward applicable to S, where S is the consequence and S_i's are the premises:

$$\frac{S_1 \quad S_2 \quad \cdots \quad S_n}{S}\,R$$

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$$\frac{S_1 \quad S_2 \quad \cdots \quad S_n}{S} R$$

For $\forall^{\circ}_{m} pS$ consider the following definition:

$$\forall^{\circ}_{m} pS = \begin{cases} \neg\Box\neg\forall^{\circ} p(\varphi \Rightarrow) & S = (\Box\varphi \Rightarrow) \\ \Box\forall^{\circ} p(\Rightarrow \psi) & S = (\Rightarrow \Box\psi) \\ \bot & \text{otherwise} \end{cases}$$

Proof Sketch for M

By induction hypothesis, $(var), (i), (ii)$ hold for all sequents T lower than S.

(var) also holds for $\forall^{\circ} pS$;

(i) To show $\Gamma, \forall^{\circ} pS \Rightarrow \Delta$ is derivable in G, it is enough to show the provability of each disjunct in G. By induction hypothesis, a similar claim holds for each $\forall^{\circ} pS_i$.

(ii) by induction on the length of the proof of $\Gamma, \Sigma \Rightarrow \Delta, \Lambda$; we take the last rule used in the proof. If it is an axiom $(\forall^{\circ}_{ax} pS)$ is used; if it is a rule in **G3w**, then $\bigvee_R (\bigwedge_i \forall^{\circ} pS_i)$ is used and if it is a modal rule $(\forall^{\circ}_{m} pS)$ is used.

Thank you!