From Kochen-Specker to Feder-Vardi

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A confluence of ideas

- Constraint satisfaction as a computational paradigm
- Contextuality in quantum mechanics and beyond
- sheaves and presheaves
- sheaf cohomology
- logic, finite model theory and descriptive complexity

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Connecting:

- concrete and abstract
- structural and algorithmic

Background

- Abramsky and Brandenburger (2011) developed a sheaf-theoretic approach to contextuality and non-locality
- Abramsky, Barbosa and Mansfield (2011) and Abramsky, Barbosa, Kishida, Lal and Mansfield (2015) developed cohomological characterisations of contextuality
- Abramsky and Dawar have a joint project on Resources and Coresources studying the interplay between structural ideas and algorithmic and complexity issues ("Structure meets Power")
- Dawar's student Adam O' Conghaile proposes (2021) a very interesting way of connecting these apparently very different topics
- Leading to ongoing joint work

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The Feder-Vardi Conjecture (1993):

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This conjecture was recently proved by Bulatov and Zhuk (*c.* 2017).

Escaping the Turing tarpit

- The CSP paradigm has some *structure*
- Can use tools e.g. from universal algebra
- Classification of templates by their *polymorphisms* (*i.e.* "symmetries")
- If the template B has only trivial symmetries, CSP(B) is NP-complete.
- If there is a non-trivial symmetry (e.g. a weak near-unanimity polymorphism), it is polynomial-time solvable.

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This is subject to the following conditions:

- **down-closure**: If $f : C \to B \in S$ and $C' \subseteq C$, then $f|_{C'} : C' \to B \in S$.
- **forth condition**: If $f : C \to B \in S$, $|C| < k$, and $a \in A$, then for some $f' : C \cup \{a\} \to B \in S$, $f'|_C = f$.

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Notation: $A \to_k B$.

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It also has a logical characterisation: $A \to_k B$ iff every k-variable existential positive FO formula satisfied by A is satisfied by B.

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- This is the *presheaf of partial homomorphisms*.
- A subpresheaf of $\mathcal{H}_k$ is a presheaf $\mathcal{S}$ such that $\mathcal{S}(C) \subseteq \mathcal{H}_k(C)$ for all $C \in \Sigma_k(A)$, and moreover if $C' \subseteq C$ and $h \in \mathcal{S}(C)$, then $\rho^C_{C'}(h) \in \mathcal{S}(C')$.

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- A subpresheaf of $\mathcal{H}_k$ is a presheaf $\mathcal{S}$ such that $\mathcal{S}(C) \subseteq \mathcal{H}_k(C)$ for all $C \in \Sigma_k(A)$, and moreover if $C' \subseteq C$ and $h \in \mathcal{S}(C)$, then $\rho^C_{C'}(h) \in \mathcal{S}(C')$.
- A presheaf is *flasque* (or "flabby") if the restriction maps are surjective. This means that if $C \subseteq C'$, each $h \in \mathcal{S}(C)$ has an extension $h' \in \mathcal{S}(C')$ with $h'|_C = h$.

Proposition

There is a bijective correspondence between

1. *positional strategies from A to B*
2. *flasque sub-presheaves of $\mathcal{H}_k$.*

Proof.

The property of being a subpresheaf of $\mathcal{H}_k$ is equivalent to the down-closure property, while being flasque is equivalent to the forth condition. □

Local consistency as coflasquification

Seen from the sheaf-theoretic perspective, the local consistency algorithm has a strikingly simple and direct mathematical specification.

Given a category $\mathcal{C}$, we write $\hat{\mathcal{C}}$ for the category of presheaves on $\mathcal{C}$. We write $\hat{\mathcal{C}}_{\mathsf{fl}}$ for the full subcategory of flasque presheaves.

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Proposition

The inclusion $\hat{\mathcal{C}}_{\mathsf{fl}} \hookrightarrow \hat{\mathcal{C}}$ has a right adjoint, so the flasque presheaves form a coreflective subcategory. The associated idempotent comonad on $\widehat{\Sigma_k(A)}$ is written as $\mathcal{S} \mapsto \mathcal{S}^{\diamond}$, where $\mathcal{S}^{\diamond}$ is the largest flasque subpresheaf of $\mathcal{S}$. The counit is the inclusion $\mathcal{S}^{\diamond} \hookrightarrow \mathcal{S}$, and idempotence holds since $\mathcal{S}^{\diamond\diamond} = \mathcal{S}^{\diamond}$. We have $\mathcal{H}_k^{\diamond} = \overline{\mathcal{S}}_k$.

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Proof.

For existence, the empty presheaf is flasque, and flasque subpresheaves are closed under unions, *i.e.* joins in the subobject lattice $\mathsf{Sub}(\mathcal{S})$.

The key point for showing couniversality is that the image of a flasque presheaf under a natural transformation is flasque. Thus any natural transformation $\mathcal{S}' \Longrightarrow \mathcal{S}$ from a flasque presheaf $\mathcal{S}'$ factors through the counit inclusion $\mathcal{S}^{\diamond} \hookrightarrow \mathcal{S}$. □

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This construction is dual to a standard construction in sheaf theory, which constructs a flasque sheaf extending a given sheaf, leading to a monad, the *Godement construction*.

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This construction is dual to a standard construction in sheaf theory, which constructs a flasque sheaf extending a given sheaf, leading to a monad, the *Godement construction*.

The following proposition shows how this comonad propagates *local inconsistency* to *global inconsistency*.

Proposition

Let $\mathcal{S}$ be a presheaf on $\Sigma_k(A)$. If $\mathcal{S}(C) = \varnothing$ for any $C \in \Sigma_k(A) \setminus \{\varnothing\}$, then $\mathcal{S}^{\diamond} = \varnothing$.

Global sections and compatible families

A global section of a flasque subpresheaf $\mathcal{S}$ of $\mathcal{H}_k$ is a natural transformation $1 \Longrightarrow \mathcal{S}$. More explicitly, it is a family $\{h_C\}_{C \in \Sigma_k(A)}$ with $h_C \in \mathcal{S}(C)$ such that, whenever $C \subseteq C'$, $h_C = h_{C'}|_C$.

Proposition

Suppose that $k \geq n$, where n is the maximum arity of any relation in σ. There is a bijective correspondence between

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Let $\mathcal{M}_k(A)$ be the maximal elements of $\Sigma_k(A)$, *i.e.* the k-element subsets. A *k-compatible family* in $\overline{\mathcal{S}}_k$ is a family $\{h_C\}_{C \in \mathcal{M}_k(A)}$ such that, for all $C, C' \in \mathcal{M}_k(A)$,

$$\rho^{C}_{C \cap C'}(h_C) = \rho^{C'}_{C \cap C'}(h'_C).$$

Proposition

There is a bijective correspondence between global sections and k-compatible families of $\overline{\mathcal{S}}_k$.

We can summarize our results so far as follows:

Proposition

There is a polynomial-time reduction from $\mathrm{CSP}(B)$ *to the problem, given any instance* A*, of determining whether the associated presheaf* $\overline{\mathcal{S}}_k$ *has a global section, or equivalently, a* k*-compatible family.*

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This will make substantial use of the prior work on contextuality, as mentioned previously.

Illustration: local consistency

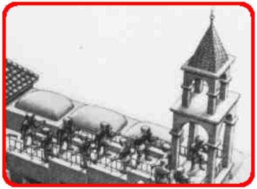
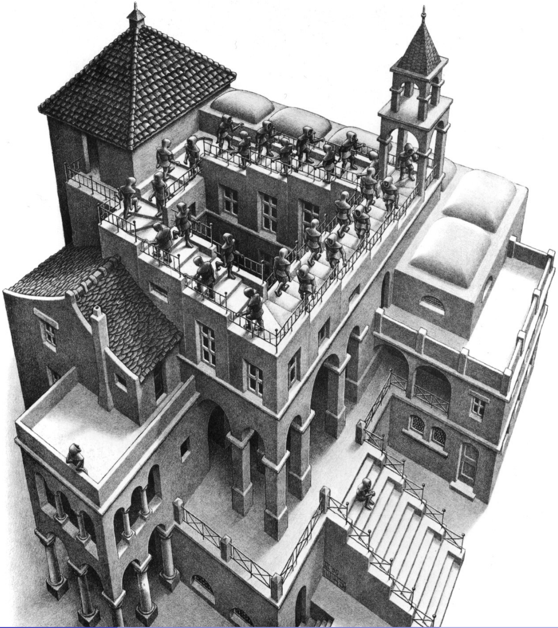


Illustration: global inconsistency



Topology of Paradox

- Clearly, the staircase *as a whole* cannot exist in the real world. Nonetheless, the constituent parts make sense *locally*.
- Quantum contextuality shows that the logical structure of quantum mechanics exhibits exactly these features of *local consistency*, but *global inconsistency*.
- We note that Escher's work was inspired by the *Penrose stairs*.
- Indeed, these figures provide more than a mere analogy. Penrose has studied the topological "twisting" in these figures using cohomology. This is quite analogous to our use of sheaf cohomology to capture the logical twisting in contextuality.
- Recent cross-over of these ideas into Constraint Satisfaction and structure isomorphism (refinements of Weisfeiler-Leman).

Adjunctions recalled

Given a ring R, the category of R-modules is denoted R–Mod. There is an evident forgetful functor $U : R\text{–Mod} \rightarrow \text{Set}$, and an adjunction

$$\text{Set} \underset{U}{\overset{F_R}{\rightleftarrows}} R\text{–Mod} \qquad (F_R \dashv U)$$

$F_R(X)$ is the free module generated by X (formal finite R-linear combinations over X).

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Note that $\mathbb{Z}$–Mod is isomorphic to AbGrp, the category of abelian groups.

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- Given A with associated presheaf $\overline{\mathcal{S}}_k$, we can define the AbGrp-valued presheaf $F_{\mathbb{Z}}\overline{\mathcal{S}}_k$.
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- Given a flasque subpresheaf $\mathcal{S}$ of $\mathcal{H}_k$, we have the AbGrp-valued presheaf $\mathcal{F} = F_{\mathbb{Z}}\mathcal{S}$. We use the Čech cohomology with respect to the cover $\mathcal{M} = \mathcal{M}_k(A)$.

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- In order to focus attention at the context $C \in \mathcal{M}$, we use the presheaf $\mathcal{F}|_C$, which "projects" onto C. The cohomology of this presheaf is the *relative cohomology* of $\mathcal{F}$ at C. The i'th relative Čech cohomology group of $\mathcal{F}$ is written as $\check{H}^i(\mathcal{M}, \mathcal{F}|_C)$.

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- The cohomological obstruction $\gamma : \mathcal{F}(C) \to \check{H}^1(\mathcal{M}, \mathcal{F}|_C)$ defined in ABKLM is this connecting homomorphism, composed with the isomorphism $\mathcal{F}(C) \cong \check{H}^0(\mathcal{M}, \mathcal{F}|_C)$.

Using the invariant

We use the following from ABKLM:

Proposition

For a local section $s \in \overline{\mathcal{S}}_k(C_0)$*, with* $C_0 \in \mathcal{M}_k(A)$*, the following are equivalent:*

1. $\gamma(s) = 0$
2. *There is a* $\mathbb{Z}$*-compatible family* $\{\alpha_C\}_{C \in \mathcal{M}_k(A)}$ *with* $\alpha_C \in F_{\mathbb{Z}}\overline{\mathcal{S}}_k(C)$*, such that, for all* $C, C' \in \mathcal{M}_k(A)$*:* $\rho^C_{C \cap C'}(\alpha_C) = \rho^{C'}_{C \cap C'}(\alpha_{C'})$*. Moreover,* $\alpha_{C_0} = 1 \cdot s$*.*

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We call a family as in (2) a $\mathbb{Z}$*-compatible extension* of s.

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We use the following from ABKLM:

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For a local section $s \in \overline{\mathcal{S}}_k(C_0)$, with $C_0 \in \mathcal{M}_k(A)$, the following are equivalent:

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We call a family as in (2) a *$\mathbb{Z}$-compatible extension* of s.

We can regard such an extension as a "$\mathbb{Z}$-linear approximation" to a homomorphism $h : A \to B$ extending s.

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Given a flasque subpresheaf $\mathcal{S}$ of $\mathcal{H}_k$, and $s \in \mathcal{S}(C)$, $C \in \mathcal{M}_k(A)$, we write $\mathbb{Z}\mathrm{ext}_k(\mathcal{S}, s)$ for the predicate which holds iff s has a $\mathbb{Z}$-compatible extension in $\mathcal{S}$.

Computing the invariant

The idea from (AOC 2021) is to use this invariant as the key ingredient in an algorithm refining k-consistency.

Proposition

There is a polynomial-time algorithm for deciding the predicate $\mathbb{Z}\mathrm{ext}_k(\mathcal{S}, s)$.

Proof.

Each constraint $\rho^{C}_{C\cap C'}(\alpha_C) = \rho^{C'}_{C\cap C'}(\alpha_{C'})$ can be written as a set of homogeneous linear equations: for each $s \in \overline{\mathcal{S}}_k(C \cap C')$, we have the equation

$$\sum_{\substack{t \in \overline{\mathcal{S}}_k(C), \\ t|_{C\cap C'} = s}} r_{C,t} \quad - \quad \sum_{\substack{t' \in \overline{\mathcal{S}}_k(C'), \\ t'|_{C\cap C'} = s}} r_{C',t'} \; = \; 0$$

in the variables $r_{C,s}$ as C ranges over contexts, and s over $\overline{\mathcal{S}}_k(C)$.
The whole system is of size polynomial in $|A|$, $|B|$. $\mathbb{Z}\mathrm{ext}_k(\mathcal{S}, s)$ is equivalent to the existence of a solution for this system of equations. Since solving systems of linear equations over $\mathbb{Z}$ is in PTIME, this yields the result. □

Cohomological k-consistency

We use this predicate as a filter to refine local consistency.

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We define $\mathcal{S}^{\Box} \hookrightarrow \mathcal{S}$ by

$$\mathcal{S}^{\Box}(C) := \begin{cases} \{s \in \mathcal{S}(C) \mid \mathbb{Z}\mathrm{ext}_k(\mathcal{S}, s)\} & C \in \mathcal{M}_k(A) \\ \mathcal{S}(C) & \text{otherwise} \end{cases}$$

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$\mathcal{S}^{\square}$ is closed under restriction, hence a presheaf. It is not necessarily flasque. Thus we are led to the following iterative process:

$$\mathcal{H}_k \hookleftarrow \mathcal{H}_k^{\diamond} \hookleftarrow \mathcal{H}_k^{\diamond\square\diamond} \hookleftarrow \cdots \hookleftarrow \mathcal{H}_k^{\diamond(\square\diamond)^m} \hookleftarrow \cdots$$

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We write $\mathcal{S}_k^{(m)}$ for the m'th iteration of this process, and $\mathcal{S}_k^*$ for the fixpoint. Note that $\overline{\mathcal{S}}_k = \mathcal{S}_k^{(0)}$.

Cohomological k-consistency and CSP

Returning to the CSP decision problem, we define some relations on structures:

- We define $A \to_k B$ iff A is *strongly k-consistent* with respect to B, *i.e.* iff $\overline{\mathcal{S}}_k = \mathcal{S}_k^{(0)} \neq \varnothing$.
- We define $A \to_k^{\mathbb{Z}} B$ if $\mathcal{S}_k^* \neq \varnothing$, and say that A is *cohomologically k-consistent* with respect to B.
- We define $A \to_k^{\mathbb{Z}(1)} B$ if $\mathcal{S}_k^{(1)} \neq \varnothing$, and say that A is *one-step cohomologically k-consistent* with respect to B.

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We can regard these relations as approximations to the "true" homomorphism relation $A \to B$. The soundness of these approximations is stated as follows:

Proposition

We have the following chain of implications:

$$A \to B \;\Rightarrow\; A \to_k^{\mathbb{Z}} B \;\Rightarrow\; A \to_k^{\mathbb{Z}(1)} B \;\Rightarrow\; A \to_k B.$$

Affine templates: the power of one iteration

We now consider the case where the template structure B is *affine*. This means that $B = R$ is a finite ring, and the interpretation of each relation in σ on R has the form

$$E^R_{\vec{a},b}(r_1, \ldots, r_n) \;\equiv\; \sum_{i=1}^{n} a_i r_i = b$$

for some $\vec{a} \in R^n$ and $b \in R$.
Thus we can label each relation in σ as $E_{\vec{a},b}$, where $\vec{a}$, b correspond to the interpretation of the relation in R.

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We say that a function $f : A \to R$ *satisfies* this equation if $\sum_{i=1}^n a_i f(x_i) = b$ holds in R, *i.e.* if $E^A_{\vec{a},b}(f(x_1), \ldots, f(x_n))$.

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It is then immediate that a function $f : A \to R$ simultaneously satisfies all the equations in $\mathbb{T}^A$ iff it is a homomorphism.

Cohomological k-consistency is exact for affine templates

We can now state an important result from AOC 2021: that cohomological k-consistency is an *exact condition* for affine templates. Moreover, the key step in the argument is the main result from ABKLM (2015), that AvN_R implies CSC.

Proposition

For every linear template R, and instance A:

$$A \to R \iff A \to_k^{\mathbb{Z}} R \iff A \to_k^{\mathbb{Z}(1)} R.$$

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The two tractable classes identified by Feder and Vardi appeared to be quite different in character.
Cohomological k-consistency captures the tractability of both!

The main question

For each template structure B, either CSP(B) is NP-complete, or B admits a weak near-unanimity polymorphism.

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A positive answer to this question would give an alternative proof of the Dichotomy Theorem.

Note that Zhuk's algorithm (and all others in this genre) makes explicit use of the polymorphism, whereas cohomological k-consistency is completely general, and applies to any CSP.

Further Developments

- The same ideas can be adapted to give a very similar analysis for the widely studied Weisfeiler-Leman equivalences, which give polynomial-time approximations to graph and structure isomorphism.
- Cohomological refinements of these equivalences can then be introduced, and are shown to defeat various families of counter-examples based on the Cai-Furer-Immerman construction, which is paradigmatic in finite model theory.

People









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