Advantages and challenges posed by PNmatrices

PNmatrix = Partial non-deterministic matrix

$\mathbb{B} = \langle \{0,1\}, \{1\}, \cdot_{\mathbb{B}} \rangle$

$\to_{\mathbb{B}}$	0	1
0	1	1
1	0	1

impose $p \to (\neg p \to \neg q)$ ⟶

$\mathbb{B}_{\mathsf{Ax}} = \langle \{00, 01, 10, 11\}, \{10, 11\}, \cdot_{\mathbb{B}_{\mathsf{Ax}}} \rangle$

$\to_{\mathbb{B}_{\mathsf{Ax}}}$	00	01	10	11
00	10	10	10	$\emptyset$
01	10	10, 11	10	11
10	00, 01	00, 01	10	$\emptyset$
11	$\emptyset$	01	$\emptyset$	11

	$\neg_{\mathbb{B}_{\mathsf{Ax}}}$
00	00, 01
01	10, 11
10	00, 01
11	11

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Plan

Logics and their combination

- Tarskian consequence relations: single-conclusion **set** $\times$ **fmla**
- Scottian consequence relations: multiple-conclusion **set** $\times$ **set**
- Posetal categories $\mathbf{Mult}$ and $\mathbf{Sing}$
- Motivation for PNmatrices: modular semantics for combined logics

Semantics: Generalized truth-functionality

- Bivaluations and categories $\mathbf{Biv}$ and $\mathbf{Biv}^{\cap}$ (isomorphic to $\mathbf{Mult}^{op}$ and $\mathbf{Sing}^{op}$)
- Semantical units: from matrices to PNmatrices
- Categories $\mathbf{PNmatr}$ and $\mathbf{SPNmatr}$ and their posetal quotients $\mathbf{Rexp}$ and $\mathbf{SRexp}$
- Galois connection between $\mathbf{Rexp}$ and $\mathbf{SRexp}$ and $\mathbf{Mult}^{op}$ and $\mathbf{Sing}^{op}$

Strict morphisms and quotients of PNmatrices, what is new?

Basic concepts

signatures

Σ: $\mathbb{N}_0$-indexed set of connectives

$\Sigma_1 \cap \Sigma_2 = \{\Sigma_1^{(n)} \cap \Sigma_2{}^{(n)}\}_{n\in\mathbb{N}_0}$

$\Sigma_1 \cup \Sigma_2 = \{\Sigma_1^{(n)} \cup \Sigma_2{}^{(n)}\}_{n\in\mathbb{N}_0}$

$\Sigma_1 \setminus \Sigma_2 = \{\Sigma_1^{(n)} \setminus \Sigma_2{}^{(n)}\}_{n\in\mathbb{N}_0}$

Propositional languages

$L = L_\Sigma(P)$ given by $\psi ::= P \mid ©(\psi, \dots, \psi)$ for $© \in \Sigma$

substitutions

$\sigma : P \to L$, $\varphi(\vec{\psi}) = \varphi(\vec{p})^\sigma$ when $\sigma(\vec{p}) = \vec{\psi}$

single-conclusion rules

set $\times$ fmla

$\dfrac{\Gamma}{\varphi}$ with $\Gamma, \{\varphi\} \subseteq L$

multiple-conclusion rules

set $\times$ set

$\dfrac{\Gamma}{\Delta}$ with $\Gamma, \Delta \subseteq L$

Notion of logic

Multiple-conclusion consequence relation

as proposed by Scott and Shoesmith&Smiley in the 70's

Internalizes case analysis

Reasoning = From certain premise-set one reaches a conclusion-set

Language ($\boldsymbol{L}$) = Set of formulas ($\varphi, \psi, \delta, \gamma, \eta, \xi, \ldots$)

$\boldsymbol{\Gamma}$ = premise-set $\qquad$ $\boldsymbol{\Delta}$ = conclusion-set

We write $\boldsymbol{\Gamma} \rhd \boldsymbol{\Delta}$ to say:

from $\boldsymbol{\Gamma}$ we conclude $\boldsymbol{\Delta}$ or

$\boldsymbol{\Delta}$ is a consequence of $\boldsymbol{\Gamma}$ or

$\boldsymbol{\Delta}$ follows from $\boldsymbol{\Gamma}$

Single- and multiple-conclusion logics

A Scottian consequence relation (**set** $\times$ **set**-cr) is a $\rhd \subseteq \wp(L) \times \wp(L)$ satisfying:

(**O**) $\Gamma \rhd \Delta$ if $\Gamma \cap \Delta \neq \emptyset$ (*overlap*)

(**D**) $\Gamma \cup \Gamma' \rhd \Delta \cup \Delta'$ if $\Gamma \rhd \Delta$ (*dilution*)

(**C**) $\Gamma \rhd \Delta$ if $\Gamma \cup \Omega \rhd \overline{\Omega} \cup \Delta'$ for every partition $\langle \Omega, \overline{\Omega} \rangle$ of some $\Theta \subseteq L$ (*cut for sets*)

(**S**) $\Gamma^\sigma \rhd \Delta^\sigma$ for any substitution $\sigma : P \to L$ if $\Gamma \rhd \Delta$ (*substitution invariance*)

Given a **set** $\times$ **set**-cr $\rhd$, its single conclusion fragment $\vdash_\rhd = \rhd \cap (\wp(L) \times L)$ is a Tarskian consequence relation (**set** $\times$ **set**-cr) satisfying:

(**R**) $\Gamma \vdash \varphi$ if $\varphi \in \Gamma$ (*reflexivity*),

(**M**) $\Gamma \cup \Gamma' \vdash \varphi$ if $\Gamma \vdash \varphi$ (*monotonicity*),

(**T**) $\Gamma \vdash \varphi$ if $\Delta \vdash \varphi$ and $\Gamma \vdash \psi$ for every $\psi \in \Delta$ (*transitivity*)

(**S**) $\Gamma^\sigma \vdash \varphi^\sigma$ for any substitution $\sigma : P \to L$ if $\Gamma \vdash \varphi$ (*substitution invariance*)

- A set of **set** $\times$ **set**-rules R is a basis for $\rhd_R$, the smallest **set** $\times$ **set**-cr containing R.
- A set of **set** $\times$ **fmla**-rules R is a basis for $\vdash_R$, the smallest **set** $\times$ **fmla**-cr containing R.

Categories Sing and Mult

Let $\mathbf{Mult}$ and $\mathbf{Sing}$ be the posetal categories where the objects are consequence relations of the correspondent type and are ordered by inclusion:

$\mathbf{Mult}$ Objects: $\langle \mathbf{\Sigma}, \rhd \rangle$ where $\rhd$ is a **set** $\times$ **set**-cr

Morphisms: $\langle \mathbf{\Sigma_1}, \rhd_\mathbf{1} \rangle \sqsubseteq \langle \mathbf{\Sigma_2}, \rhd_\mathbf{2} \rangle$ if $\mathbf{\Sigma_1} \subseteq \mathbf{\Sigma_2}$ and $\rhd_\mathbf{1} \subseteq \rhd_\mathbf{2}$

$\mathbf{Sing}$ Objects: $\langle \mathbf{\Sigma}, \vdash \rangle$ where $\vdash$ is a **set** $\times$ **fmla**-cr

Morphisms: $\langle \mathbf{\Sigma_1}, \vdash_\mathbf{1} \rangle \sqsubseteq \langle \mathbf{\Sigma_2}, \vdash_\mathbf{2} \rangle$ if $\mathbf{\Sigma_1} \subseteq \mathbf{\Sigma_2}$ and $\vdash_\mathbf{1} \subseteq \vdash_\mathbf{2}$

Facts:

- Both are complete lattices.
- $\mathbf{Sing}$ is embeddable in $\mathbf{Mult}$ by sending $\langle \mathbf{\Sigma}, \vdash \rangle$ to $\langle \mathbf{\Sigma}, \rhd_\vdash \rangle$

  where $\rhd_\vdash$ is the smallest **set** $\times$ **set**-cr such that $\vdash \subseteq \rhd$.

  That is,

  $$\mathbf{\Gamma} \rhd_\vdash \mathbf{\Delta} \text{ iff there is } \boldsymbol{\delta} \in \mathbf{\Delta} \text{ such that } \mathbf{\Gamma} \vdash \boldsymbol{\delta}$$
- $\mathbf{Sing}$ is a full subcategory of $\mathbf{Mult}$
- For $\propto \in \{\rhd, \vdash\}$

  $$\mathbf{type}(\propto) = \begin{cases} \textbf{set} \times \textbf{set} & \text{if } \propto = \rhd \\ \textbf{set} \times \textbf{fmla} & \text{if } \propto = \vdash \end{cases}$$

Two standard ways of presenting a logic

Let L be a language, $\Gamma, \Delta \subseteq L$ and $\varphi \in L$

- Via semantics

  given $\mathcal{M}$ set of models as bivaluations $m : L \to \{\mathbf{true}, \mathbf{false}\}$

  - Multiple-conclusion (**set** $\times$ **set**)

    $\Gamma \rhd_{\mathcal{M}} \Delta :=$ If for every $m \in \mathcal{M}$, $m(\Gamma) = \{\mathbf{true}\}$ then $\mathbf{true} \in m(\Delta)$
  - Single-conclusion (**set** $\times$ **set**)

    $\Gamma \vdash_{\mathcal{M}} \varphi :=$ If for every $m \in \mathcal{M}$, $m(\Gamma) = \{\mathbf{true}\}$ then $m(\varphi) = \mathbf{true}$

  Dual reading: Premise-set (conjuntive) and conclusion-set (disjunctive).

- Via deductive systems

  - Multiple-conclusion (**set** $\times$ **set**)

    Given set of $R \subseteq \wp(L) \times \wp(L)$ of **set** $\times$ **set**-rules

    $\Gamma \rhd_R \varphi$ if there is a proof of Δ from Γ

    R axiomatizes $\rhd$
  - Single-conclusion (**set** $\times$ **set**)

    Given set of $R \subseteq \wp(L) \times L$ of **set** $\times$ **set**-rules

    $\Gamma \vdash_R \varphi$ if there is a proof of φ from Γ using the rules in R

    R axiomatizes $\vdash$

  Axiomatizations as basis for the corresponding notion of logic

Multiple-conclusion calculi and tree-proofs

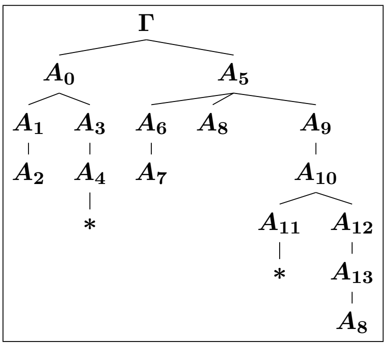
A *calculus* is a set of rules (schema) $R \subseteq \wp(L) \times \wp(L)$.

Proofs can be arboreal as rules with a conclusion set with more than a formula impose branching (case split).

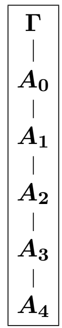
Rules with empty conclusion set discontinues the branches where there are applied.

We write $\Gamma \rhd_R \Delta$ if there is a proof starting with Γ and having a formula of Δ in each non-discontinued branch.

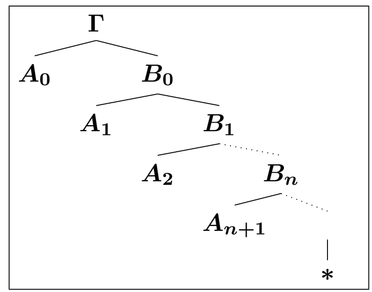
$\Gamma \rhd_R A_2, A_7, A_8$



$\Gamma \rhd_R A_4$



$\Gamma \rhd_R \{A_i : i \in \mathbb{N}\}$



$\rhd_R$ is the smallest **set** $\times$ **set**-cr containing R,

R is a proper basis for $\rhd_R$

If R are all **set** $\times$ **fmla** then $\rhd_R = \rhd_{\vdash_R}$.

Combining logics by joining their calculi: fibring

Being complete lattices both $\mathbf{Mult}$ and $\mathbf{Sing}$ have joins and meets.

Given two logics $\langle \Sigma_1, \propto_1 \rangle$ and $\langle \Sigma_2, \propto_2 \rangle$ of the same type, either **set** $\times$ **set** or **set** $\times$ **fmla**, (either both in $\mathbf{Mult}$ or both in $\mathbf{Sing}$)

Their join is

$$\langle \Sigma_1, \propto_1 \rangle \sqcup \langle \Sigma_2, \propto_2 \rangle = \langle \Sigma_1 \cup \Sigma_2, \propto_1 \bullet \propto_2 \rangle$$

where $\propto_1 \bullet \propto_2$ is the smallest cr of the same type over $L_{\Sigma_1 \cup \Sigma_2}(P)$ containing $\propto_1$ and $\propto_2$.

Their meet is simply

$$\langle \Sigma_1, \propto_1 \rangle \sqcap \langle \Sigma_2, \propto_2 \rangle = \langle \Sigma_1 \cap \Sigma_2, \propto_1 \cap \propto_2 \rangle$$

Facts:

Given two sets of rules of the appropriate type R_1 and R_2 we have that

$$\propto_{R_1} \bullet \propto_{R_2} = \propto_{R_1 \cup R_2}$$

Examples of combining logics by joining their calculi

- Language extensions

Adding new connectives to a logic without imposing anything about them

Given $\rhd$ and $\vdash$ over $\boldsymbol{\Sigma_0} \subseteq \boldsymbol{\Sigma}$ let $\rhd^{\boldsymbol{\Sigma}}$ and $\vdash^{\boldsymbol{\Sigma}}$ over $\boldsymbol{\Sigma}$

$\boldsymbol{\Gamma} \rhd^{\boldsymbol{\Sigma}} \boldsymbol{\Delta}$ iff

$\boldsymbol{\Gamma_0} \rhd \boldsymbol{\Delta_0}$ for some $\boldsymbol{\Gamma_0} \subseteq \boldsymbol{L_{\Sigma_0}(P)}$, $\boldsymbol{\Delta_0} \subseteq \boldsymbol{L_{\Sigma_0}(P)}$, $\boldsymbol{\sigma : P \to L_{\Sigma}(P)}$ with $\boldsymbol{\Gamma_0^\sigma} \subseteq \boldsymbol{\Gamma}$, $\boldsymbol{\Delta_0^\sigma} \subseteq \boldsymbol{\Delta}$

- Fusion of modal logics

Seminal example and well understood via gluing Kripke frames for each of the combined logic.

- Combining classical AND and OR

Let $\boldsymbol{R_{\wedge\vee}}$ be formed by the **set** $\times$ **set**-rules

$$\frac{p \wedge q}{p} \qquad \frac{p \wedge q}{q} \qquad \frac{p}{p \vee q} \qquad \frac{p \vee p}{p}$$

$$\frac{p \wedge q}{q} \qquad \frac{p \quad q}{p \wedge q} \qquad \frac{p \vee q}{q \vee p} \qquad \frac{p \vee (q \vee r)}{(p \vee r) \vee q}$$

Can we combine the semantics of the $\wedge$ and $\vee$ fragments of Boolean classical matrix into a semantics for $\vdash_{\boldsymbol{R_{\wedge\vee}}}$?

Semantics of bivaluations

Susko thesis (1977), models are bivaluations $\mathbf{0} = \mathbf{true}$ and $\mathbf{1} = \mathbf{false}$

$$\textbf{Bival}(L) = \{b : L \to \{0,1\}\}$$

Given $\mathbf{B} \subseteq \textbf{Bival}(L)$ closed for substitutions let

$\Gamma \rhd_{\mathbf{B}} \Delta$ if and only if there is no $b \in \mathbf{B}$ such that $b(\Gamma) \subseteq \{1\}$ and $b(\Delta) \subseteq \{0\}$.

- $\rhd_{\mathbf{B}}$ is a **set** $\times$ **set**-cr
- $\vdash_{\mathbf{B}} = \vdash_{\rhd_{\mathbf{B}}}$ is a **set** $\times$ **set**-fmla

Ultimately, the various kinds of semantics (matrices, Kripke/neighborhood frames, relational/topological structures) are just clever ways of presenting bivaluations exploring the algebraic structure of the language.

Bivaluations in set $\times$ set and set $\times$ fmla

Given $\{b_i : i \in I\} \subseteq \mathbf{B}$ their intersection is

$$\bigcap_{i \in I} \{b_i : i \in I\} = b$$

where $b(A) = 1$ iff $b_i(A) = 1$ for every $i \in I$.

Let $\mathbf{B}^{\cap}$ denote the closure for intersections of bivaluations in $\mathbf{B}$.

Facts: $\rhd_{\mathbf{B_1}} = \rhd_{\mathbf{B_2}}$ iff $\mathbf{B_1} = \mathbf{B_2}$ and $\vdash_{\mathbf{B_1}} = \vdash_{\mathbf{B_2}}$ iff $\mathbf{B}_1^{\cap} = \mathbf{B}_2^{\cap}$

Given $\Sigma_0 \subseteq \Sigma$ and $\mathbf{B} \subseteq \mathbf{Bival}(L_{\Sigma_0}(P))$, let $\mathbf{B}^{\Sigma} = \{b \circ \mathbf{skel}_{\Sigma_0} : b \in \mathbf{B}\}$ is closed for substitutions.

Facts: $\rhd_{\mathbf{B}^{\Sigma}} = \rhd_{\mathbf{B}}^{\Sigma}$ and $\vdash_{\mathbf{B}^{\Sigma}} = \vdash_{\mathbf{B}}^{\Sigma}$

$\mathbf{skel}_{\Sigma_0}$ is a bijection between $L_{\Sigma_0}(P)$ and $L_{\Sigma}(P)$ capturing the view of an arbitrary $L_{\Sigma}(P)$ formula from the point of view of Σ_0 The idea is simply to replace $\Sigma \setminus \Sigma_0$-headed formulas by dedicated variables, just renaming the original variables.

Posetal categories $\mathbf{Biv}$ and $\mathbf{Biv}^{\cap}$

For $\mathbf{B} \subseteq \mathbf{Bival}(L_\Sigma(P))$ closed for substitutions let

$$\mathbf{Mult}(\mathbf{B}) = \langle \Sigma, \rhd_{\mathbf{B}} \rangle \quad \text{and} \quad \mathbf{Sing}(\mathbf{B}) = \langle \Sigma, \vdash_{\mathbf{B}} \rangle.$$

$\mathbf{Biv}$:

Objects: $\langle \Sigma, \mathbf{B} \rangle$ with $\mathbf{B} \subseteq \mathbf{Bival}(L_\Sigma(P))$ closed for substitutions

Morphisms: $\langle \Sigma_1, \mathbf{B}_1 \rangle \sqsubseteq \langle \Sigma_2, \mathbf{B}_2 \rangle$ iff $\Sigma_2 \subseteq \Sigma_1$ and $\mathbf{B}_1 \subseteq \mathbf{B}_2^{\Sigma_1}$.

$\mathbf{Biv}^{\cap}$:

Objects: $\langle \Sigma, \mathbf{B} \rangle$ with $\mathbf{B} \subseteq \mathbf{Bival}(L_\Sigma(P))$ closed for substitutions and intersections

Morphisms: $\langle \Sigma_1, \mathbf{B}_1 \rangle \sqsubseteq \langle \Sigma_2, \mathbf{B}_2 \rangle$ iff $\Sigma_2 \subseteq \Sigma_1$ and $\mathbf{B}_1 \subseteq \mathbf{B}_2^{\Sigma_1}$.

Facts

- $\mathrm{Mult} : \mathbf{Biv} \to \mathbf{Mult}$ is a dual order isomorphism, that is:
  Mult is bijective, and $\langle \Sigma, \mathbf{B} \rangle \sqsubseteq \langle \Sigma_0, \mathbf{B}_0 \rangle$ iff $\langle \Sigma_0, \rhd_{\mathbf{B}_0} \rangle \sqsubseteq \langle \Sigma, \rhd_{\mathbf{B}} \rangle$.
- $\mathrm{Sing} : \mathbf{Biv}^{\cap} \to \mathbf{Sing}$ is a dual order isomorphism, that is:
  Sing is bijective, and $\langle \Sigma, \mathbf{B} \rangle \sqsubseteq \langle \Sigma_0, \mathbf{B}_0 \rangle$ iff $\langle \Sigma_0, \vdash_{\mathbf{B}_0} \rangle \sqsubseteq \langle \Sigma, \vdash_{\mathbf{B}} \rangle$.
- $\langle \Sigma, \mathbf{B}_1 \rangle \sqcup \langle \Sigma, \mathbf{B}_2 \rangle = \langle \Sigma, \mathbf{B} \rangle$ where $\mathbf{B}$ is the closure for substitutions of $\mathbf{B}_1 \cup \mathbf{B}_2$
- $\langle \Sigma, \mathbf{B}_1 \rangle \sqcap \langle \Sigma, \mathbf{B}_2 \rangle = \langle \Sigma, \mathbf{B}_1 \cap \mathbf{B}_2 \rangle$
- $\mathrm{Funct}(\langle \Sigma, \mathbf{B}_1 \rangle \sqcup \langle \Sigma, \mathbf{B}_2 \rangle) = \mathrm{Funct}(\langle \Sigma, \mathbf{B}_1 \rangle) \sqcap \mathrm{Funct}(\langle \Sigma, \mathbf{B}_2 \rangle)$ for $\mathrm{Funct} \in \{\mathrm{Mult}, \mathrm{Sing}\}$
- $\mathrm{Funct}(\langle \Sigma, \mathbf{B}_1 \rangle \sqcap \langle \Sigma, \mathbf{B}_2 \rangle) = \mathrm{Funct}(\langle \Sigma, \mathbf{B}_1 \rangle) \sqcup \mathrm{Funct}(\langle \Sigma, \mathbf{B}_2 \rangle)$ for $\mathrm{Funct} \in \{\mathrm{Mult}, \mathrm{Sing}\}$

Basic examples

The smallest logic in Σ

Let $\Gamma \rhd_{\mathsf{Bival}(L_\Sigma(P))} \Delta$ iff $\Gamma \cap \Delta \neq \emptyset$

The largest logic in Σ

$\Gamma \rhd_{\mathsf{emptyset}} \Delta$ holds for every $\Gamma, \Delta \subseteq L_\Sigma(P)$

The most common general way of defining sets of bivaluations closed by substitutions is by considering logical matrices.

Logics induced by logical matrices

Given signature $\Sigma = \{\Sigma\}_{n\in\mathbb{N}}$ and fixed $L = L_\Sigma(P)$

Logical matrix $\mathbb{M} = \langle V, \cdot_{\mathbb{M}}, \mathbf{D}\rangle$

where $\langle V, \cdot_{\mathbb{M}}\rangle$ is an algebra of truth-values

set endowed with operations $©_{\mathbb{M}} : V^n \to V$ for $© \in \Sigma^{(n)}$

$\mathbf{D} \subseteq V$ is the set of designated elements corresponding to $\mathbf{1}$

Valuations over $\mathbb{M}$ are $v : L \to V$ satisfying

$$v(©(\varphi_1, \dots, \varphi_k)) = ©_{\mathbb{M}}(v(\varphi_1), \dots, v(\varphi_k))$$

Given valuation v and substitution σ, $v \circ \sigma$ is also a valuation. Where $v \circ \sigma(\varphi) = v(\varphi^\sigma)$

Each valuation induces a bivaluation

$$b_v(\varphi) = \begin{cases} \mathbf{1} & \text{if } v(\varphi) \in \mathbf{D} \\ \mathbf{0} & \text{if } v(\varphi) \notin \mathbf{D} \end{cases}$$

The set $\mathrm{BVal}(\mathbb{M}) = \mathbf{B}_{\mathbb{M}} = \{b_v : v \text{ valuation over } \mathbb{M}\}$ is closed under substitutions, since $b_v \circ \sigma = b_{v\circ\sigma}$

Let $\rhd_{\mathbb{M}} = \rhd_{\mathbf{B}_{\mathbb{M}}}$ and $\vdash_{\mathbb{M}} = \vdash_{\mathbf{B}_{\mathbb{M}}} = \vdash_{\rhd_{\mathbb{M}}}$.

$\Gamma \rhd_{\mathbb{M}} \Delta$ iff there is no valuation v over $\mathbb{M}$ such that $v(\Gamma) \subseteq D$ and $v(\varphi) \subseteq V \setminus D$.

Or equivalently, for every v over $\mathbb{M}$, $v(\Gamma) \subseteq D$ implies $v(\Delta) \cap D \neq \emptyset$.

There is no finite matrix $\mathbb{M}$ such that $\rhd_{\mathbb{M}} = \rhd_{\mathbf{Bival}(L_\Sigma(P))}$ nor $\vdash_{\mathbb{M}} = \vdash_{\mathbf{emptyset}}$!

Extending truth-functionality: non-determinism and partiality

A Σ-matrix is a tuple $\mathbb{M} = \langle V, \cdot_{\mathbb{M}}, D \rangle$

- V is a non-empty set (of *truth-values*)
- $D \subseteq V$ (the set of *designated* truth-vales)
- $©_{\mathbb{M}} : V^n \to V$ for each $c \in \Sigma^{(n)}$

Other ways of defining sets of bivaluations closed for substitution:

non-determinism A Σ-Nmatrix is ... with $©_{\mathbb{M}} : V^n \to \wp(V) \setminus \{\emptyset\}$

from Avron & Lev 2005

'Non-deterministic multiple-valued structures', JAR 2013

partiality A Σ-Pmatrix is ... with $©_{\mathbb{M}} : V^n \to \{\{a\} : a \in V\} \cup \{\emptyset\}$

both A Σ-PNmatrix is ... with $©_{\mathbb{M}} : V^n \to \wp V$

from Baaz, Lahav & Zamansky's

'Finite-valued semantics for canonical labelled calculi', JAR 2013

Valuations over $\mathbb{M}$ are $v : L \to V$ satisfying

$$v(©(\varphi_1, \ldots, \varphi_k)) \in ©_{\mathbb{M}}(v(\varphi_1), \ldots, v(\varphi_k))$$

The set $\mathrm{BVal}(\mathbb{M}) = \mathbf{B}_{\mathbb{M}} = \{b_v : v \text{ valuation over } \mathbb{M}\}$ is closed under substitutions, since $b_v \circ \sigma = b_{v \circ \sigma}$ and we let also $\rhd_{\mathbb{M}} = \rhd_{\mathbf{B}_{\mathbb{M}}}$ and $\vdash_{\mathbb{M}} = \vdash_{\mathbf{B}_{\mathbb{M}}} = \vdash_{\rhd_{\mathbb{M}}}$

PNmatrices are nice!

- Almost(!) every logic can be characterized by a single PNmatrix
  enough for signature to contain a connective of arity > 1
- PNmatrices retain many nice properties of matrices
  when finite: logic is finitary, **SAT** in NP, decision in coNP
- Many non-finitely valued logics have finite PNsemantics
- Natural semantics for logical strengthenings and combined logics
- Finite PNmatrices still can be axiomatized by analytical $\mathbf{set} \times \mathbf{set}$-calculi

Some 2-valued Nmatrices you should know

$\mathbb{M}_{\text{free}}$

$©_{\text{free}}$	0	1
0	$0,1$	$0,1$
1	$0,1$	$0,1$

$\rhd_{\mathbb{M}_{\text{free}}}$ is axiomatized by the emptyset of rules

$\mathbb{M}_{\text{mp}}$

$\to_{\text{mp}}$	0	1
0	$0,1$	$0,1$
1	0	$0,1$

$\rhd_{\mathbb{M}_{\text{mp}}}$ is axiomatized by modus ponens $\frac{p\,,\,p\to q}{q}$

$\mathbb{M}_{\text{sq}}$

	0	1
$\mathbb{M}_{\text{sq}}$	$0,1$	1

$\rhd_{\mathbb{M}_{\text{sq}}}$ is axiomatized by $\Box$-generalization $\frac{p}{\Box p}$

None of the logics induced by the Nmatrices above is induced by a finite matrix (or even by a finite set of finite matrices.

- 2^{2^n} n-ary Boolean functions and
- 3^{2^n} n-ary Boolean multi-functions (81 binary Boolean multi-functions, 16 of them are Boolean functions)
- Hence, 65 that are not functions
- Every set of Boolean multi-functions generates a different logic that can automatically be provided with an analytic axiomatization

Language extensions and non-determinism

Adding new connectives to a logic without imposing anything on them

Given Σ_0-PNmatrix $\mathbb{M} = \langle V, \cdot_{\mathbb{M}}, D \rangle$ let $\mathbb{M}^{\Sigma} = \langle V, \cdot_{\mathbb{M}^{\Sigma}}, D \rangle$ with

$$©(a_1, \ldots, a_k) = \begin{cases} ©_{\mathbb{M}}(a_1, \ldots, a_k) & \text{if } © \in \Sigma_0 \\ V & \text{otherwise} \end{cases}$$

Facts:

- $\mathrm{BVal}(\mathbb{M}^{\Sigma}) = \mathrm{BVal}(\mathbb{M})^{\Sigma}$
- $\rhd_{\mathbb{M}^{\Sigma}} = \rhd_{\mathbb{M}}^{\Sigma}$ and $\vdash_{\mathbb{M}^{\Sigma}} = \vdash_{\mathbb{M}}^{\Sigma}$
- If general, if $\Sigma \setminus \Sigma_0$ contains a 0-ary connective then there is no single matrix characterizing $\rhd^{\Sigma}$ or $\vdash^{\Sigma}$
- If general, if $\Sigma \setminus \Sigma_0$ contains a n-ary connective with $n > 0$ then there is no finite set of finite matrices characterizing $\rhd^{\Sigma}$ or $\vdash^{\Sigma}$

More examples

Matrix: $\mathbb{M}_{\mathsf{DB}} = \langle \{f, \bot, \top, t\}, \cdot_{\mathbb{M}_{\mathsf{DB}}}, \{\top, t\}\rangle$

Dunn-Belnap logic, dealing with partial information about atomic formulas

	$\neg_{\mathbb{M}_{\mathsf{DB}}}(x)$
f	t
$\bot$	$\bot$
$\top$	$\top$
t	f

$\wedge_{\mathbb{M}_{\mathsf{DB}}}$	f	$\bot$	$\top$	t
f	f	f	f	f
$\bot$	f	$\bot$	f	$\bot$
$\top$	f	f	$\top$	$\top$
t	f	$\bot$	$\top$	t

$\vee_{\mathbb{M}_{\mathsf{DB}}}$	f	$\bot$	$\top$	t
f	f	$\bot$	$\top$	t
$\bot$	$\bot$	$\bot$	t	t
$\top$	$\top$	t	$\top$	t
t	t	t	t	t

Nmatrix: $\mathbb{M}_{\mathsf{P}} = \langle \{f, \bot, \top, t\}, \cdot_{\mathbb{M}_{\mathsf{P}}}, \{\top, t\}\rangle$

Processors logic dealing with partial information about complex formulas decidable in $\mathbf{PTIME}$

	$\neg_{\mathbb{M}_{\mathsf{P}}}(x)$
f	t
$\bot$	$\bot$
$\top$	$\top$
t	f

$\wedge_{\mathbb{M}_{\mathsf{P}}}$	f	$\bot$	$\top$	t
f	f	f	f	f
$\bot$	f	$f, \bot$	f	$f, \bot$
$\top$	f	f	$\top$	$\top$
t	f	$f, \bot$	$\top$	$\top, t$

$\vee_{\mathbb{M}_{\mathsf{P}}}$	f	$\bot$	$\top$	t
f	$f, \top$	$\bot, t$	$\top$	t
$\bot$	$\bot, t$	$\bot, t$	t	t
$\top$	$\top$	t	$\top$	t
t	t	t	t	t

Pmatrix: $\mathbb{M}_{\mathsf{K}} = \langle \{0, a, b, 1\}, \cdot_{\mathbb{M}_{\mathsf{K}}}, \{b, 1\}\rangle$

Kleene of order, two matrices put together

	$\neg_{\mathbb{M}_{\mathsf{K}}}(x)$
0	1
a	a
b	b
1	0

$\wedge_{\mathbb{M}_{\mathsf{K}}}$	0	a	b	1
0	0	0	0	0
a	0	a	$\emptyset$	a
b	0	$\emptyset$	b	b
1	0	a	b	1

$\vee_{\mathbb{M}_{\mathsf{K}}}$	0	a	b	1
0	0	a	b	1
a	a	a	$\emptyset$	1
b	b	$\emptyset$	b	1
1	1	1	1	1

Categories of PNmatrices $\mathbf{PNmatr}$ and $\mathbf{Rexp}$

A function $f : V_1 \to V_2$ is a strict morphism between $\mathbb{M}_1 = \langle \Sigma_1, \cdot_{\mathbb{M}}, D_1 \rangle$ and $\mathbb{M}_2 = \langle \Sigma_2, \cdot_{\mathbb{M}}, D_2 \rangle$ if $\Sigma_2 \subseteq \Sigma_1$ and satisfies $f^{-1}(D_2) = D_1$ and for $© \in \Sigma_2^n$,

$$f(©_{\mathbb{M}_1}(x_1, \ldots, x_n)) \subseteq ©_{\mathbb{M}_2}(f(x_1), \ldots, f(x_n))$$

$\mathbf{PNmatr}$:

Objects: $\langle \Sigma, \mathbb{M} \rangle$ with $\mathbb{M}$ a PNmatrix over Σ

Morphisms: strict morphisms between PNmatrices

$\mathbf{Rexp}$: Avron called pre-images by strict homomorphisms rexpansions

Objects: $\langle \Sigma, \mathbb{M} \rangle$ with $\mathbb{M}$ a PNmatrix over Σ

Morphisms: $\langle \Sigma_1, \mathbb{M}_1 \rangle \sqsubseteq \langle \Sigma_2, \mathbb{M}_2 \rangle$ iff $\Sigma_2 \subseteq \Sigma_1$ and there is some strict morphism between $\mathbb{M}_1$ and $\mathbb{M}_2$. Equivalently, if $\mathbb{M}_1$ is a rexpansion of $\mathbb{M}_2$.

Facts:

- $\mathbf{Rexp}$ is a posetal category
- $\mathbf{Rexp}$ is the result quotientating the **hom** sets in $\mathbf{PNmatr}$ into a single element
- The quotient functor $Q : \mathbf{PNmatr} \to \mathbf{Rexp}$ is continuous and cocontinuous
- Q transforms products in meets and coproducts in joins

Saturation and the ω-power

We say a PNmatrix $\mathbb{M}$ is saturated whenever $\rhd_{\mathbb{M}} = \rhd_{\vdash_{\mathbb{M}}}$, that is, whenever for every $\Gamma \rhd_{\mathbb{M}} \Delta$ iff there is $\delta \in \Delta$ such that $\Gamma \vdash_{\mathbb{M}} \delta$.

Let $\mathbf{SPNmatr}$ and $\mathbf{SRexp}$ the full subcategories of $\mathbf{PNmatr}$ and $\mathbf{Rexp}$ where the objects are restricted to saturated PNmatrices.

Let $\mathbb{M}^\omega = \langle V^\omega, \cdot_\omega, D^\omega \rangle$ with

$$©_\omega(s_1, \ldots, s_k) = \{s \in V^\omega : s(i) \in ©_{\mathbb{M}}(s_1(i), \ldots, s_k(i))\}$$

Facts:

- $\langle \Sigma, \mathbb{M} \rangle$ is saturated if and only if $\mathrm{BVal}(\mathbb{M})^\cap = \{1\} \cup \mathrm{BVal}(\mathbb{M})$
- $\vdash_{\mathbb{M}} = \vdash_{\mathbb{M}^\omega}$
- $\rhd_{\mathbb{M}^\omega} = \rhd_{\vdash_{\mathbb{M}}}$
- The Boolean Nmatrices shown before are all saturated

Strict product of PNmatrices

Given Σ_1- and Σ_2-PNmatrices $\mathbb{M}_1 = \langle A_1, \cdot_1, D_1\rangle$ and $\mathbb{M}_2 = \langle A_2, \cdot_2, D_2\rangle$,
let $U_1 = A_1 \setminus D_1$ and $U_2 = A_2 \setminus D_2$.
Their strict product is the $\Sigma_1 \cup \Sigma_2$-PNmatrix

$$\mathbb{M}_1 \star \mathbb{M}_2 = \langle A_{12}, \cdot_{12}, D_{12}\rangle$$

where

$$A_{12} = (D_1 \times D_2) \cup (U_1 \times U_2) \qquad D_{12} = D_1 \times D_2$$

$$©_{12}((a_1, b_1), \dots, (a_k, b_k)) = \begin{cases} \{(a,b) \in A_{12} : a \in ©_1(a_1, \dots, a_k)\} & \text{if } c \in \Sigma_1 \setminus \Sigma_2 \\ \{(a,b) \in A_{12} : b \in ©_2(b_1, \dots, b_k)\} & \text{if } c \in \Sigma_2 \setminus \Sigma_1 \\ \{(a,b) \in A_{12} : a \in ©_1(a_1, \dots, a_k) & \\ \qquad \text{and } b \in ©_2(b_1, \dots, b_k)\} & \text{if } c \in \Sigma_1 \cap \Sigma_2 \end{cases}$$

Note that $©_{12}((a_1, b_1), \dots, (a_k, b_k)) = \emptyset$
if $©_1(a_1, \dots, a_k) \subseteq D_1$ and $©_2(a_1, \dots, a_k) \subseteq U_2$ or vice versa.

Consider the projection functions, i.e., $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$.

Facts about strict-product

- π_1 and π_2 are strict-morphisms
- $\mathrm{BVal}(\mathbb{M}_1 * \mathbb{M}_2) = \mathrm{BVal}(\mathbb{M}_1^{\Sigma_1 \cup \Sigma_2}) \cap \mathrm{BVal}(\mathbb{M}_2^{\Sigma_1 \cup \Sigma_2})$.
  - If $v \in \mathrm{Val}(\mathbb{M}_1 * \mathbb{M}_2)$ then $(\pi_k \circ v) \in \mathrm{Val}({\mathbb{M}_k}^{\Sigma_1 \cup \Sigma_2})$
  - $v_1 \in \mathrm{Val}(\mathbb{M}_1^{\Sigma_1 \cup \Sigma_2})$, $v_2 \in \mathrm{Val}(\mathbb{M}_2^{\Sigma_1 \cup \Sigma_2})$, and $v_1(\varphi) \in D_1$ iff $v_2(\varphi) \in D_2$ for every $A \in L_{\Sigma_1 \cup \Sigma_2}(P)$, then $v_1 * v_2 \in \mathrm{Val}(\mathbb{M}_1 * \mathbb{M}_2)$ with $v_1 * v_2(\varphi) = (v_1(\varphi), v_2(\varphi))$
- $\mathbb{M}_1 \star \mathbb{M}_2$ is saturated whenever $\mathbb{M}_1$ and $\mathbb{M}_2$ are
- $\langle \Sigma_1, \mathbb{M}_1 \rangle \otimes \langle \Sigma_2, \mathbb{M}_2 \rangle = \langle \Sigma_1 \cup \Sigma_2, \mathbb{M}_1 \star \mathbb{M}_2 \rangle$ is the product in all the introduced categories $\mathbf{PNmatr}$, $\mathbf{Rexp}$, $\mathbf{SPNmatr}$ and $\mathbf{SRexp}$.

Modular semantics for combined logics

- $\mathsf{B}_{\mathbb{M}_1 \star \mathbb{M}_2} = \mathsf{B}^{\Sigma_2}_{\mathbb{M}_1} \cap \mathsf{B}^{\Sigma_1}_{\mathbb{M}_2}$
- $\rhd_{\mathbb{M}_1} \sqcup \rhd_{\mathbb{M}_2} = \rhd_{\mathbb{M}_1 \star \mathbb{M}_2}$
- If $\mathbb{M}_1$ and $\mathbb{M}_2$ saturated then $\vdash_{\mathbb{M}_1} \sqcup \vdash_{\mathbb{M}_2} = \vdash_{\mathbb{M}_1 \star \mathbb{M}_2}$
- If either $\mathbb{M}_1$ or $\mathbb{M}_2$ not saturated it may happen that $\vdash_{\mathbb{M}_1} \sqcup \vdash_{\mathbb{M}_2} \subsetneq \vdash_{\mathbb{M}_1 \star \mathbb{M}_2}$
- $\mathsf{B}_{\mathbb{M}_1^\omega \star \mathbb{M}_2^\omega} = (\mathsf{B}^{\Sigma_2}_{\mathbb{M}_1})^\cap \cap (\mathsf{B}^{\Sigma_1}_{\mathbb{M}_2})^\cap$
- $\vdash_{\mathbb{M}_1} \sqcup \vdash_{\mathbb{M}_2} = \vdash_{\mathbb{M}_1^\omega \star \mathbb{M}_2^\omega}$

Back to combining AND and OR

$$\mathbf{CPL}_{\wedge} = \mathcal{L}_{\mathbb{M}_{\wedge}} \qquad\qquad \mathbf{CPL}_{\vee} = \mathcal{L}_{\mathbb{M}_{\vee}}$$

Let

$\mathbb{M}_{\wedge}$:

$\tilde{\wedge}$	0	1
0	0	0
1	0	1

$\mathbb{M}_{\vee}$:

$\tilde{\vee}$	0	1
0	0	1
1	1	1

$\mathbb{M}_{\wedge} \star \mathbb{M}_{\vee} = \mathbb{M}_{\wedge\vee}$ is the $\wedge\vee$-fragment of classical Boolean matrix

$\mathbb{M}_{\wedge}$ is saturated but $\mathbb{M}_{\vee}$ is not. $p \vee q \rhd_{\mathbb{M}_{\vee}} p, q$ but $p \vee q \not\vdash_{\mathbb{M}_{\vee}} p$ and $p \vee q \not\vdash_{\mathbb{M}_{\vee}} q$

$\vdash_{\wedge\vee^{\omega}} = \vdash_{\mathbb{M}_{\wedge} \star \mathbb{M}_{\vee}^{\omega}}$ and $\mathbb{M}_{\wedge\vee} \cong \mathbb{M}_{\wedge} \star \mathbb{M}_{\vee}^{\omega}$ where

$\mathbb{M}_{\wedge\vee} = \langle \wp(\mathbb{N}), \cdot_{\#}, \{\mathbb{N}\} \rangle$ with $X \vee_{\#} Y = X \cup Y$ and

$$X \wedge_{\#} Y = \begin{cases} \mathbb{N} & \text{if } X = Y = \mathbb{N} \\ \wp(\mathbb{N}) & \text{otherwise} \end{cases}$$

Facts:

- There is no **set** $\times$ **fmla**-axiomatization of classical logic by gathering the axiomatization of the fragments with a single connective
- Classical logic can be **set** $\times$ **set**-axiomatized by joining the axiomatizations for each of the connectives

Partiality allows for a badly behaved sum

Let $\mathcal{M} = \{\langle \Sigma, \mathbb{M}_i \rangle : i \in I\}$ be a set of PNmatrices, each $\mathbb{M}_i = \langle V_i, D_i, \cdot_{\mathbb{M}_i} \rangle$. The *sum* of $\mathcal{M}$ is the PNmatrix $(\Sigma, \oplus\mathcal{M})$ where $\oplus\mathcal{M} = \langle V, D, \cdot_{\oplus} \rangle$ and

$$V = \bigcup_{i \in I} (\{i\} \times V_i)$$

$$D = \bigcup_{i \in I} (\{i\} \times D_i)$$

$$©_{\oplus}((i_1, x_1), \ldots, (i_n, x_n)) = \begin{cases} \{i\} \times ©_{\mathbb{M}_i}(x_1, \ldots, x_n)) & \text{if } i = i_1 = \cdots = i_n \\ \emptyset & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}_0$ and $c \in \Sigma^{(n)}$.

$(\Sigma, \oplus\mathcal{M})$ is a coproduct of $\mathcal{M}$ in $\mathbf{PNMatr}$, with inclusion homomorphisms

$$\iota_i : \langle \Sigma, \mathbb{M}_i \rangle \to \langle \Sigma, \oplus\mathcal{M} \rangle$$

defined, for each $i \in I$ and each $x \in V_i$, by $\iota_i(x) = (i, x)$.

Hence,

$$\bigcup_{i \in I} \mathrm{BVal}(\mathbb{M}_i) \subseteq \mathrm{BVal}(\oplus\mathcal{M})$$

Perhaps surprisingly, however, it may happen that $\mathrm{BVal}(\oplus\mathcal{M}) \neq \bigcup_{i \in I} \mathrm{BVal}(\mathbb{M}_i)$.

A sufficient condition for the equality to hold is that the Σ contains at least a connective with arity > 1.

Gathering the Lindenbaum bundle into a Pmatrix

For $\mathbf{\Gamma} \subseteq \boldsymbol{L}_{\mathbf{\Sigma}}(\boldsymbol{P})$, let $\mathbb{M}_{\mathbf{\Gamma}} = \langle \boldsymbol{L}_{\mathbf{\Sigma}}(\boldsymbol{P}), \cdot, \mathbf{\Gamma} \rangle$.

Lindenbaum bundle

$\mathrm{Lind}(\langle \mathbf{\Sigma}, \rhd \rangle) = \{\mathbb{M}_{\mathbf{\Gamma}} : \mathbf{\Gamma} \not\rhd (\boldsymbol{L}_{\mathbf{\Sigma}}(\boldsymbol{P}) \setminus \mathbf{\Gamma})\}$

Lindenbaum Pmatrix

Let

$$\mathrm{Lind}_{\oplus}(\langle \mathbf{\Sigma}, \rhd \rangle) := \bigoplus \mathrm{Lind}(\langle \mathbf{\Sigma}, \rhd \rangle)$$

and for **set** $\times$ **fmla**-cr $\vdash$

$$\mathrm{Lind}_{\oplus}(\langle \mathbf{\Sigma}, \vdash \rangle) := \bigoplus \mathrm{Lind}(\langle \mathbf{\Sigma}, \rhd_{\vdash} \rangle)$$

Galois connection between Rexp and Biv

Consider the functors, in this case, also lattice morphisms

$\mathrm{BVal} : \mathbf{Rexp} \to \mathbf{Biv}$ such that $\mathrm{BVal}(\langle \Sigma, \mathbb{M} \rangle) = \langle \boldsymbol{\Sigma}, \mathrm{BVal}(\mathbb{M}) \rangle$

$\mathrm{Lind}_{\oplus} : \mathbf{Biv} \to \mathbf{Rexp}$ by $\mathrm{Lind}_{\oplus}(\langle \boldsymbol{\Sigma}, \mathbf{B} \rangle) = \langle \boldsymbol{\Sigma}, \bigoplus \mathrm{Lind}(\mathbf{B}) \rangle$

$\mathrm{BVal}^{+1} : \mathbf{SRexp} \to \mathbf{Biv}^{\cap}$ such that $\mathrm{BVal}^{+1}(\langle \Sigma, \mathbb{M} \rangle) = \langle \boldsymbol{\Sigma}, \{1\} \cup \mathrm{BVal}(\mathbb{M}) \rangle$

$\mathrm{Lind}_{\oplus}^{-1} : \mathbf{Biv}^{\cap} \to \mathbf{SRexp}$ by $\mathrm{Lind}_{\oplus}^{-1}(\langle \boldsymbol{\Sigma}, \mathbf{B} \rangle) = \langle \boldsymbol{\Sigma}, \bigoplus \mathrm{Lind}(\mathbf{B} \setminus \{1\}) \rangle$

Facts:

- The functors $\mathrm{Lind}_{\oplus}, \mathrm{BVal}$ constitute a Galois connection, that is, for every $\langle \boldsymbol{\Sigma}, \mathbf{B} \rangle$ in $\mathbf{Biv}$ and every $\langle \boldsymbol{\Sigma_0}, \mathbb{M}_0 \rangle$ in $\mathbf{Rexp}$:

  $$\mathrm{Lind}_{\oplus}(\langle \boldsymbol{\Sigma}, \mathbf{B} \rangle) \sqsubseteq \langle \boldsymbol{\Sigma_0}, \mathbb{M}_0 \rangle \quad \text{iff} \quad \langle \boldsymbol{\Sigma}, \mathbf{B} \rangle \sqsubseteq \mathrm{BVal}(\langle \boldsymbol{\Sigma_0}, \mathbb{M}_0 \rangle)$$

- The functors $\mathrm{Lind}_{\oplus}^{-1}, \mathrm{BVal}^{+1}$ constitute a Galois connection, that is, for every $\langle \boldsymbol{\Sigma}, \mathbf{B} \rangle$ in $\mathbf{Biv}^{\cap}$ and every $\langle \boldsymbol{\Sigma_0}, \mathbb{M}_0 \rangle$ in $\mathbf{SRexp}$:

  $$\mathrm{Lind}_{\oplus}^{-1}(\langle \boldsymbol{\Sigma}, \mathbf{B} \rangle) \sqsubseteq \langle \boldsymbol{\Sigma_0}, \mathbb{M}_0 \rangle \quad \text{iff} \quad \langle \boldsymbol{\Sigma}, \mathbf{B} \rangle \sqsubseteq \mathrm{BVal}^{+1}(\langle \boldsymbol{\Sigma_0}, \mathbb{M}_0 \rangle)$$

Categorical view

$$\mathbf{PNMatr} \xrightarrow{\mathcal{Q}} \mathbf{Rexp} \underset{\mathrm{Lind}_{\oplus}}{\overset{\mathrm{BVal}}{\rightleftarrows}} \mathbf{Biv} \xrightarrow[\cong]{\mathrm{Mult}} \mathbf{Mult}^{\mathbf{op}} \qquad (\mathrm{BVal} \top \mathrm{Lind}_{\oplus})$$

$$\mathbf{SPNMatr} \xrightarrow{\mathcal{Q}} \mathbf{SRexp} \underset{\mathrm{Lind}_{\oplus}^{-1}}{\overset{\mathrm{BVal}^{+1}}{\rightleftarrows}} \mathbf{Biv}^{\cap} \xrightarrow[\cong]{\mathrm{Sing}} \mathbf{Sing}^{\mathbf{op}} \qquad (\mathrm{BVal}^{+1} \top \mathrm{Lind}_{\oplus}^{-1})$$

- Is there Adjunction between $\mathbf{PNmatr}$ and $\mathbf{Mult}$? How to associate a logic with a PNmatrix such that there is a unique morphism to every PNmatrix characterizing a weaker logic?

- $\mathbf{Rexp}$ deals with unicity but in $\mathbf{Rexp}$ but the existent of strict morphisms is clearly insufficient to detect if PNmatrices define the same logic, and the kernels of Sing and Mult much more complex on PNmatrices than in matrices.

Problems $\triangleright_{\mathbb{M}_1} \overset{?}{=} \triangleright_{\mathbb{M}_2}$ and $\vdash_{\mathbb{M}_1} \overset{?}{=} \vdash_{\mathbb{M}_2}$

Example

	$\neg_{\mathbb{M}_1}(x)$
0	1
1	0
T	$0, T$

	$\neg_{\mathbb{M}_2}(x)$
0	1
1	0
T	$1, T$

	$\neg_{\mathbb{M}_3}(x)$
0	1
1	0
T	$0, 1, T$

	$\neg_{\mathbb{M}_4}(x)$
0	1
1	0
T	$0, T$
T'	$1, T$

Facts:

- $\mathrm{BVal}(\mathbb{M}_1) = \mathrm{BVal}(\mathbb{M}_2) = \mathrm{BVal}(\mathbb{M}_3) = \mathrm{BVal}(\mathbb{M}_4)$
- $\triangleright_{\mathbb{M}_1} = \triangleright_{\mathbb{M}_2} = \triangleright_{\mathbb{M}_3} = \triangleright_{\mathbb{M}_4}$ and $\vdash_{\mathbb{M}_1} = \vdash_{\mathbb{M}_2} = \vdash_{\mathbb{M}_3} = \vdash_{\mathbb{M}_4}$
- $\mathbb{M}_1 \sqsubseteq \mathbb{M}_3$, $\mathbb{M}_2 \sqsubseteq \mathbb{M}_3$
- $\mathbb{M}_1 \not\sqsubseteq \mathbb{M}_2$, $\mathbb{M}_2 \not\sqsubseteq \mathbb{M}_1$ and $\mathbb{M}_3 \not\sqsubseteq \mathbb{M}_4$
- $\mathbb{M}_4 \sqsubseteq \mathbb{M}_3$ and $\mathbb{M}_3$ is a quotient of $\mathbb{M}_4$.

Furthermore, the problem of, given arbitrary finite (P)Nmatrices the problem $\vdash_{\mathbb{M}_1} \overset{?}{=} \vdash_{\mathbb{M}_2}$ is undecidable.

In the multiple-conclusion setting it is still open but we suspect that the same holds for deciding $\triangleright_{\mathbb{M}_1} = \triangleright_{\mathbb{M}_2}$.

What changes regarding strict morphisms and quotients

Over matrices

- Kernels of strict morphisms between matrices are congruences compatible with the set of designated elements and surjective strict morphisms (and quotients) preserve the logic (both single and multiple)
- For finite reduced $\mathbf{\Sigma}$-matrices $\mathbb{M}_1$ and $\mathbb{M}_2$ $\rhd_{\mathbb{M}_1} = \rhd_{\mathbb{M}_2}$ IFF there are strict morphisms $f_{12} : \mathbb{M}_1 \to \mathbb{M}_2$ and $f_{21} : \mathbb{M}_2 \to \mathbb{M}_1$ (Shoesmith and Smiley 1978)

Over PNmatrices

- Any quotient of a PNmatrix by an equivalence relation compatible with the set of designated values is still a PNmatrix and induces a strict morphism (and vice-versa)
- A strict (surjective or not) morphism $f : \mathbb{M}_1 \to \mathbb{M}_2$ only implies that $\rhd_{\mathbb{M}_2} \subseteq \rhd_{\mathbb{M}_1}$
- Strict morphisms (and quotients) of PNmatrices may generate stronger logics
- Of course that if there are strict morphisms $f_{12} : \mathbb{M}_1 \to \mathbb{M}_2$ and $f_{21} : \mathbb{M}_2 \to \mathbb{M}_1$ then $\rhd_{\mathbb{M}_1} = \rhd_{\mathbb{M}_2}$ but the other direction fails
- Perhaps a local explanation for $\rhd_{\mathbb{M}_1} = \rhd_{\mathbb{M}_2}$ soundness is not possible

Some applications of PNmatrices and strict morphisms

• There is a general recipe that generates semantics for axiomatic extensions by pre-images of the original semantics, yielding

- a denumerable (but quite syntactic) semantics for axiomatic extensions of logics with denumerable PNmatrix semantics (including intuitionistic and every modal logics, remember that modus ponnens and generalization can be captured by a 2-valued Nmatrix)
- a finiteness preserving semantics for a wide range of base logics and axioms satisfying certain shapes

• Going back to Avron's logic for processors dealing with partial informations from various sources, by coding it in a finite PNmatrix and using the algorithm generating analytical **set** $\times$ **set** -axiomatization we discovered that this logic was decidable in **PTIME** since the generated rules are all of type **set** $\times$ **fmla** (no branching needed)

$\wedge_{\mathbb{S}}$	f	$\bot$	$\top$	t
f	f	f	f	f
$\bot$	f	$f, \bot$	f	$f, \bot$
$\top$	f	f	$\top$	$\top$
t	f	$f, \bot$	$\top$	$t, \top$

$\vee_{\mathbb{S}}$	f	$\bot$	$\top$	t
f	$f, \top$	$t, \bot$	$\top$	t
$\bot$	$t, \bot$	$t, \bot$	t	t
$\top$	$\top$	t	$\top$	t
t	t	t	t	t

	$\neg_{\mathbb{S}}$
f	t
$\bot$	$\bot$
$\top$	$\top$
t	f

$$\frac{p\ ,\ q}{p \wedge q}\ r_1 \qquad \frac{p \wedge q}{p}\ r_2 \qquad \frac{p \wedge q}{q}\ r_3 \qquad \frac{\neg p}{\neg(p \wedge q)}\ r_4 \qquad \frac{\neg q}{\neg(p \wedge q)}\ r_5$$

$$\frac{p}{p \vee q}\ r_6 \qquad \frac{q}{p \vee q}\ r_7 \qquad \frac{\neg(p \vee q)}{\neg p}\ r_8 \qquad \frac{\neg(p \vee q)}{\neg q}\ r_9 \qquad \frac{\neg p\ ,\ \neg q}{\neg(p \vee q)}\ r_{10}$$

$$\frac{p}{\neg\neg p}\ r_{11} \qquad \frac{\neg\neg p}{p}\ r_{12}$$

Bibliography

Non-referenced facts and examples were taken from:

- **Axiomatizing non-deterministic many-valued generalized CRs**
  Synthese (Caleiro & M. 2019)
- **Analytic calculi for monadic PNmatrices**
  WoLLIC (Caleiro & M. 2019)
- **On axioms and rexpansions**
  Book chapter, OCL dedicated to Arnon Avron, (Caleiro & M. 2020)
- **Computational properties of partial non-deterministic logical matrices**
  LFCS (Caleiro, Filipe & M. 2021)
- **Comparing logics induced by partial non-deterministic semantics**
  In preparation (Caleiro, Filipe & M.)

Extra slide for full circle: A proper PNmatrix

The strengthening of the logic of classical implication with $\boldsymbol{p \to (\neg p \to \neg q)}$ is characterized by $\mathbb{M} = \langle \{\mathbf{00}, \mathbf{01}, \mathbf{10}, \mathbf{11}\}, \{\mathbf{10}, \mathbf{11}\}, \cdot_{\mathbb{M}} \rangle$ with

$\to_{\mathbb{M}}$	$\mathbf{00}$	$\mathbf{01}$	$\mathbf{10}$	$\mathbf{11}$
$\mathbf{00}$	$\mathbf{10}$	$\mathbf{10}$	$\mathbf{10}$	$\emptyset$
$\mathbf{01}$	$\mathbf{10}$	$\mathbf{10}, \mathbf{11}$	$\mathbf{10}$	$\mathbf{11}$
$\mathbf{10}$	$\mathbf{00}, \mathbf{01}$	$\mathbf{00}, \mathbf{01}$	$\mathbf{10}$	$\emptyset$
$\mathbf{11}$	$\emptyset$	$\mathbf{01}$	$\emptyset$	$\mathbf{11}$

	$\neg_{\mathbb{M}}$
$\mathbf{00}$	$\mathbf{00}, \mathbf{01}$
$\mathbf{01}$	$\mathbf{10}, \mathbf{11}$
$\mathbf{10}$	$\mathbf{00}, \mathbf{01}$
$\mathbf{11}$	$\mathbf{11}$

Maximal sub-Nmatrices:

$\mathbb{M}_{\{\mathbf{00},\mathbf{01},\mathbf{10}\}} = \langle \{\mathbf{00}, \mathbf{01}, \mathbf{10}\}, \{\mathbf{10}\}, \cdot_{\mathbb{M}} \rangle$ $\qquad$ $\mathbb{M}_{\{\mathbf{01},\mathbf{11}\}} = \langle \{\mathbf{00}, \mathbf{11}\}, \{\mathbf{11}\}, \cdot_{\mathbb{M}} \rangle$

$\to_{\mathbb{M}}$	$\mathbf{00}$	$\mathbf{01}$	$\mathbf{10}$	$\neg_{\mathbb{M}}$
$\mathbf{00}$	$\mathbf{10}$	$\mathbf{10}$	$\mathbf{10}$	$\mathbf{00}, \mathbf{01}$
$\mathbf{01}$	$\mathbf{10}$	$\mathbf{10}, \mathbf{11}$	$\mathbf{10}$	$\mathbf{10}, \mathbf{11}$
$\mathbf{10}$	$\mathbf{00}, \mathbf{01}$	$\mathbf{00}, \mathbf{01}$	$\mathbf{10}$	$\mathbf{00}, \mathbf{01}$

$\to_{\mathbb{M}^\sharp_{\mathsf{Ax}}}$	$\mathbf{01}$	$\mathbf{11}$	$\neg_{\mathbb{M}}$
$\mathbf{01}$	$\mathbf{11}$	$\mathbf{11}$	$\mathbf{11}$
$\mathbf{11}$	$\mathbf{01}$	$\mathbf{11}$	$\mathbf{11}$

$\rhd_{\mathbb{M}}$ is not characterizable by any finite set of finite matrices!

PNmatrices are very maleable semantics for compositional semantics