Semilinear idempotent distributive ℓ-monoids

Simon Santschi

Mathematical Institute
University of Bern

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Motivation

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- Semilinear idempotent distributive ℓ-monoids.
- A structure theorem for the finite subdirectly irreducibles.
- A description of the subvariety lattice and a proof that it is countably infinite.

Distributive ℓ-monoids

A **distributive ℓ-monoid** is an algebra $\langle M, \wedge, \vee, \cdot, e\rangle$ such that

- $\langle M, \wedge, \vee\rangle$ is a distributive lattice,
- $\langle M, \cdot, e\rangle$ is a monoid,
- for all $a, b, c, d \in M$

$$a(b \wedge c)d = abd \wedge acd \quad \text{and} \quad a(b \vee c)d = abd \vee acd.$$

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The class $\mathcal{DLM}$ of all distributive ℓ-monoids forms a variety (equational class). Similarly, the classes $\mathcal{I}d\mathcal{DLM}$ and $\mathcal{CI}d\mathcal{DLM}$ of idempotent and commutative idempotent distributive ℓ-monoids are varieties.

Example

The inverse-free reduct of an ℓ-group is a distributive ℓ-monoid.

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Example

A totally ordered monoid $\langle M, \cdot, e, \leq \rangle$, i.e., a monoid $\langle M, \cdot, e \rangle$ with a total order $\leq$ on M such that $a \leq b$ implies $cad \leq cbd$ for $a, b, c, d \in M$, can be considered as a distributive ℓ-monoid $\langle M, \min, \max, \cdot, e \rangle$ which we also call totally ordered monoid.

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Corollary

The variety $\mathcal{CI}d\mathcal{DLM}$ is the subvariety of $\mathcal{S}em\mathcal{I}d\mathcal{DLM}$ consisting of the commutative members of $\mathcal{S}em\mathcal{I}d\mathcal{DLM}$.

Local finiteness of $\mathcal{I}d\mathcal{DLM}$

Recall that an algebra $\mathbf{A}$ is called locally finite if every finitely generated subalgebra of $\mathbf{A}$ is finite and a class $\mathcal{K}$ of algebras is called locally finite if every member of $\mathcal{K}$ is locally finite.

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Using this fact we can show:

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Corollary

$\mathcal{S}em\mathcal{I}d\mathcal{DLM}$ is locally finite and generated by the class of finite subdirectly irreducible totally ordered idempotent monoids.

Examples

The totally ordered idempotent monoids $\mathbf{C}_2$, $\mathbf{C}_2^\partial$, $\mathbf{G}_3$, and $\mathbf{D}_3$ described below are exactly the simple totally ordered idempotent monoids:

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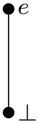
$\cdot$	e	$\bot$
e	e	$\bot$
$\bot$	$\bot$	$\bot$



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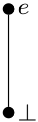

$\mathbf{C}_2^\partial = \langle \{e, \top\}, \cdot, e, \leq \rangle$

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e	e	$\top$
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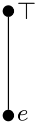

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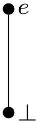

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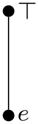
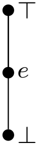


$$\mathbf{G}_3 = \langle\{\bot, e, \top\}, \cdot, e, \leq\rangle$$

$\cdot$	e	$\bot$	$\top$
e	e	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$



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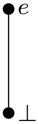

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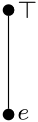

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$\cdot$	e	$\bot$	$\top$
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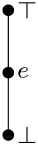

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$\cdot$	e	$\bot$	$\top$
e	e	$\bot$	$\top$
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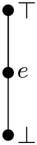
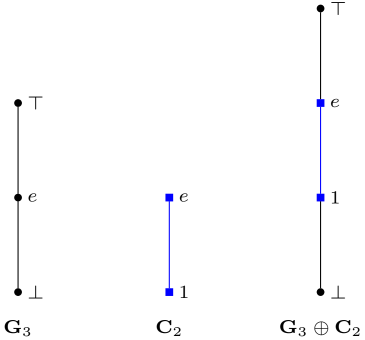


Combining totally ordered idempotent monoids

$\cdot$	e	1	$\bot$	$\top$
e	e	1	$\bot$	$\top$
1	1	1	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$



The e-sum construction

The **e-sum** (also known as **nested sum** or $\mathbf{K}[\mathbf{L}]$ in (Galatos 2004)) was used in (Olson 2012) to study the subvariety lattice of the variety of semilinear idempotent residuated lattices.

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Intuition: $\mathbf{L} \oplus \mathbf{M}$ is obtained by replacing the identity e in $\mathbf{L}$ with $\mathbf{M}$ and extending the order and product in such a way that the elements of $\mathbf{M}$ behave like e with respect to elements of $\mathbf{L}$.

Properties of the e-sum

Lemma (cf. Olson 2012)

Let $\mathbf{L}$ and $\mathbf{M}$ be totally ordered idempotent monoids. Then $\mathbf{L} \oplus \mathbf{M}$ is an totally ordered idempotent monoid. Moreover, $\mathbf{L}$ and $\mathbf{M}$ embed into $\mathbf{L} \oplus \mathbf{M}$ via the inclusion maps.

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Lemma (Olson 2012)

Let $\mathbf{L}$, $\mathbf{M}$, *and* $\mathbf{N}$ *be totally ordered idempotent monoids. Then*

$$\mathbf{L} \oplus (\mathbf{M} \oplus \mathbf{N}) \cong (\mathbf{L} \oplus \mathbf{M}) \oplus \mathbf{N}.$$

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So it makes sense to write $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_n$ for totally ordered idempotent monoids $\mathbf{M}_1, \ldots, \mathbf{M}_n$ or shorter $\bigoplus_{i=1}^{n} \mathbf{M}_i$, where $\bigoplus_{i=1}^{0} \mathbf{M}_i = \mathbf{0}$ for some fixed trivial algebra $\mathbf{0}$. We note that $\mathbf{M} \oplus \mathbf{0} \cong \mathbf{0} \oplus \mathbf{M} \cong \mathbf{M}$ for all $\mathbf{M}$, i.e., $\mathbf{0}$ is the neutral element of the e-sum operation.

Back to the Examples

The totally ordered idempotent monoids $\mathbf{C}_2$, $\mathbf{C}_2^\partial$, $\mathbf{G}_3$, and $\mathbf{D}_3$ are indecomposable with respect to the e-sum:

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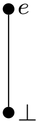

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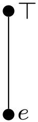

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$\cdot$	e	$\bot$	$\top$
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$\bot$	$\bot$	$\bot$	$\bot$
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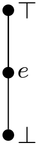

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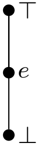

The e-sum decomposition

Theorem

Every finite totally ordered idempotent monoid $\mathbf{M}$ *is isomorphic to an* e*-sum* $\bigoplus_{i=1}^{n} \mathbf{M}_i$ *with* $\mathbf{M}_i \in \{\mathbf{C}_2, \mathbf{C}_2^{\partial}, \mathbf{G}_3, \mathbf{D}_3\}$*. Moreover, this* e*-sum is unique with respect to the algebras* $\{\mathbf{C}_2, \mathbf{C}_2^{\partial}, \mathbf{G}_3, \mathbf{D}_3\}$*.*

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Corollary (cf. Gil-Férez, Jipsen, Metcalfe 2020)

The number $\mathbf{I}(n)$ *of totally ordered idempotent monoids with* $n \in \mathbb{N} \setminus \{0\}$ *elements (up to isomorphism) is recursively defined by* $\mathbf{I}(1) = 1$*,* $\mathbf{I}(2) = 2$*, and*

$$\mathbf{I}(n) = 2 \cdot \mathbf{I}(n-1) + 2 \cdot \mathbf{I}(n-2) \quad (n > 2).$$

Moreover,

$$\mathbf{I}(n) = \frac{(1+\sqrt{3})^n - (1-\sqrt{3})^n}{2\sqrt{3}}.$$

Finite subdirectly irreducibles of $\mathcal{SemIdDLM}$

Theorem

Let $\mathbf{M}$ be a non-trivial finite semilinear idempotent distributive ℓ-monoid. Then the following are equivalent:

1. $\mathbf{M}$ *is subdirectly irreducible.*
2. $\mathbf{M} \cong \bigoplus_{i=1}^{n} \mathbf{M}_i$ *for some* $n \in \mathbb{N} \setminus \{0\}$ *with* $\mathbf{M}_i \in \{\mathbf{C}_2, \mathbf{C}_2^{\partial}, \mathbf{G}_3, \mathbf{D}_3\}$ *such that if* $\mathbf{M}_i = \mathbf{M}_{i+1}$, *then* $\mathbf{M}_i \in \{\mathbf{G}_3, \mathbf{D}_3\}$ *for every* $i \in \{1, \ldots, n-1\}$.
3. $\mathbf{Con}(\mathbf{M})$ *is a chain.*

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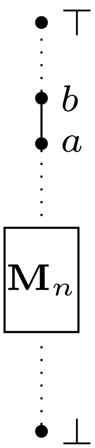
2. $\mathbf{M} \cong \bigoplus_{i=1}^{n} \mathbf{M}_i$ *for some* $n \in \mathbb{N} \setminus \{0\}$ *with* $\mathbf{M}_i \in \{\mathbf{C}_2, \mathbf{C}_2^{\partial}, \mathbf{G}_3, \mathbf{D}_3\}$ *such that if* $\mathbf{M}_i = \mathbf{M}_{i+1}$*, then* $\mathbf{M}_i \in \{\mathbf{G}_3, \mathbf{D}_3\}$ *for every* $i \in \{1, \dots, n-1\}$.
3. $\mathbf{Con}(\mathbf{M})$ *is a chain.*

Corollary

The number $\mathbf{S}(n)$ *of subdirectly irreducible totally ordered idempotent monoids with* $n \in \mathbb{N} \setminus \{0\}$ *elements (up to isomorphism) is recursively defined by* $\mathbf{S}(1) = 1$, $\mathbf{S}(2) = 2$, $\mathbf{S}(3) = 4$, *and*

$$\mathbf{S}(n) = \mathbf{S}(n-1) + 2\mathbf{S}(n-2) + 2\mathbf{S}(n-3) \quad (n > 3).$$

Ilustration of the irreducibility condition



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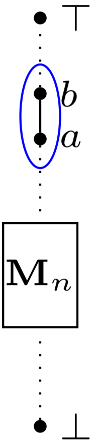

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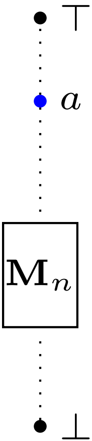

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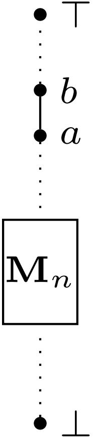

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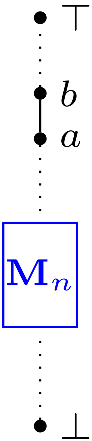

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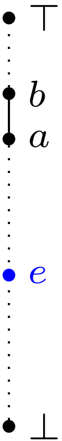

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Describing the subvariety lattice

Goal: Describe the subvariety lattice of $\mathcal{S}em\mathcal{I}d\mathcal{DLM}$.

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- For algebras $\mathbf{A}$ and $\mathbf{B}$ we define the relation $\leq_{HS}$ ($\leq_{IS}$) by

$$\mathbf{A} \leq_{HS} \mathbf{B} \text{ iff } \mathbf{A} \in HS(\{\mathbf{B}\}) \; (\mathbf{A} \leq_{IS} \mathbf{B} \text{ iff } \mathbf{A} \in IS(\{\mathbf{B}\})),$$

  where H, S, and I denote the closure under homomorphic images, subalgebras and ismorphic images.

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  where H, S, and I denote the closure under homomorphic images, subalgebras and ismorphic images.
- For a variety $\mathcal{V}$ of finite type we denote by $\mathcal{V}_*$ a fixed set which contains (up to isomorphism) exactly one copy of each finite subdirectly irreducible of $\mathcal{V}$. Then $\langle \mathcal{V}_*, \leq_{HS} \rangle$ and $\langle \mathcal{V}_*, \leq_{IS} \rangle$ are partially ordered sets.

A theorem about subvariety lattices

Theorem (Davey 1979)

Let $\mathcal{V}$ be a congruence-distributive, locally finite variety of finite type. Then the subvariety lattice of $\mathcal{V}$ is completely distributive and is isomorphic to the lattice of order ideals of the poset $\langle \mathcal{V}_*, \leq_{HS} \rangle$.

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Corollary (Olson 2012)

For a a congruence-distributive, locally finite variety $\mathcal{V}$ of finite type the subvariety lattice of $\mathcal{V}$ is countable iff $\langle \mathcal{V}_*, \leq_{HS} \rangle$ is a well quasi-ordered set.

The order $\leq_{HS}$ on $\mathcal{S}em\mathcal{I}d\mathcal{DLM}_*$

We fix $\mathcal{S}em\mathcal{I}d\mathcal{DLM}_*$ to be the set of finite subdirectly irreducibles of $\mathcal{S}em\mathcal{I}d\mathcal{DLM}$ that are e-sums of the algebras $\mathbf{C}_2, \mathbf{C}_2^\partial, \mathbf{G}_3, \mathbf{D}_3$.

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Lemma

Every homomorphic image of an totally ordered idempotent monoid $\mathbf{M}$ is isomorphic to a subalgebra of $\mathbf{M}$.

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So it suffices to show that $\langle \mathcal{S}em\mathcal{I}d\mathcal{DLM}_*, \leq_{IS} \rangle$ is a well quasi-ordered set.

Higman's Lemma

For a quasi-ordered set $\langle P, \leq \rangle$ we define the order $\leq^*$ on the set $\sigma(P)$ of finite sequences of P by

$$\langle p_1, \ldots, p_n \rangle \leq^* \langle q_1, \ldots, q_m \rangle :\Longleftrightarrow \text{ there exists an order embedding } f\colon \{1, \ldots, n\} \to \{1, \ldots, m\} \text{ such that } p_i \leq q_{f(i)} \text{ for all } i \in \{1, \ldots, n\}.$$

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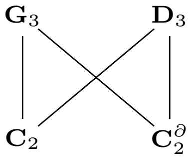
Lemma (Higman 1952)

If $\langle P, \leq \rangle$ is a well quasi-ordered set, then so is $\langle \sigma(P), \leq^* \rangle$.

The order $\leq_{IS}$ restricted to $\{\mathbf{C}_2, \mathbf{C}_2^\partial, \mathbf{G}_3, \mathbf{D}_3\}$

The order $\leq_{IS}$ restricts to the following well quasi-order on the set $\{\mathbf{C}_2, \mathbf{C}_2^\partial, \mathbf{G}_3, \mathbf{D}_3\}$.



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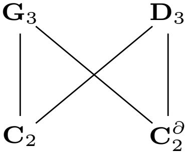

We define for $\mathbf{L}, \mathbf{M} \in \mathcal{S}em\mathcal{I}d\mathcal{DLM}_*$ the relation $\preceq$ by

$$\mathbf{L} \preceq \mathbf{M} :\Longleftrightarrow \langle \mathbf{L}_1, \ldots, \mathbf{L}_m \rangle \leq_{IS}^* \langle \mathbf{M}_1, \ldots, \mathbf{M}_n \rangle,$$

for $\mathbf{L} = \mathbf{L}_1 \oplus \cdots \oplus \mathbf{L}_m$, $\mathbf{M} = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_n$ with $\mathbf{M}_i, \mathbf{L}_j \in \{\mathbf{C}_2, \mathbf{C}_2^\partial, \mathbf{G}_3, \mathbf{D}_3\}$.

The subvariety lattice of $\mathcal{S}em\mathcal{I}d\mathcal{DLM}$

Using Higman's Lemma and the fact that the restriction of a well quasi-order to a subset is again a well quasi-order, we obtain:

Corollary

$\langle \mathcal{S}em\mathcal{I}d\mathcal{DLM}_*, \preceq \rangle$ *is a well quasi-ordered set.*

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For all $\mathbf{L}, \mathbf{M} \in \mathcal{S}em\mathcal{I}d\mathcal{DLM}_*$ *we have* $\mathbf{L} \preceq \mathbf{M}$ *if and only if* $\mathbf{L} \leq_{IS} \mathbf{M}$.

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Thus we get that $\langle \mathcal{S}em\mathcal{I}d\mathcal{DLM}_*, \leq_{IS} \rangle$ is a well quasi-ordered set, yielding

Theorem

The subvariety lattice of $\mathcal{S}em\mathcal{I}d\mathcal{DLM}$ is countably infinite.

A description of the subvariety lattice of $\mathcal{S}em\mathcal{I}d\mathcal{DLM}$

- Using the theorem of (Davey 1979) and the fact that $\leq_{HS} = \preceq$ we get that the subvariety lattice of $\mathcal{S}em\mathcal{I}d\mathcal{DLM}$ is isomorphic to the lattice of order-ideals of $\langle \mathcal{S}em\mathcal{I}d\mathcal{DLM}_*, \preceq \rangle$, via the map that maps an order ideal $\mathcal{I}$ to the variety $V(\mathcal{I})$ generated by $\mathcal{I}$.

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- Thus, by the characterization of the finite subdirectly irreducibles and the definition of $\preceq$ via the Higman order, we get a description of the subvariety lattice of $\mathcal{S}em\mathcal{I}d\mathcal{DLM}$.

For $\mathcal{CI}d\mathcal{DLM}$ the previous theorems yield the following immediate results:

Proposition

Let $\mathbf{M}$ *be a finite commutative totally ordered idempotent monoid. Then* $\mathbf{M} \cong \bigoplus_{i=1}^{n} \mathbf{M}_i$ *with* $\mathbf{M}_i \in \{\mathbf{C}_2, \mathbf{C}_2^{\partial}\}$*. Moreover* $\mathbf{M}$ *is subdirectly irreducible if and only if for all* $i \in \{1, \ldots, n-1\}$, $\mathbf{M}_i \neq \mathbf{M}_{i+1}$.

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Corollary

There are up to isomorphism 2^{n-1} *totally ordered commutative idempotent monoids of size* $n \geq 1$.

Subdirectly irreducibles of $\mathcal{CI}d\mathcal{DLM}$

For $n > 2$ we define inductively the algebras $\mathbf{C}_n$ and $\mathbf{C}_n^\partial$ by

$$\mathbf{C}_n := \mathbf{C}_2 \oplus \mathbf{C}_{n-1}^\partial$$
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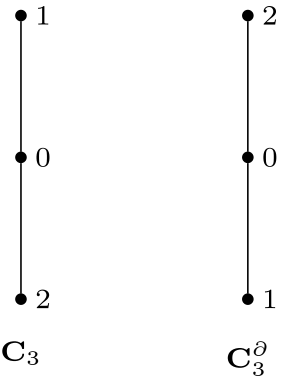
Proposition

For every $n > 1$ the algebras $\mathbf{C}_n$ and $\mathbf{C}_n^\partial$ are up to isomorphism the only subdirectly irreducible totally ordered commutative idempotent monoids with n elements.

Example $\mathbf{C}_3$ and $\mathbf{C}_3^\partial$

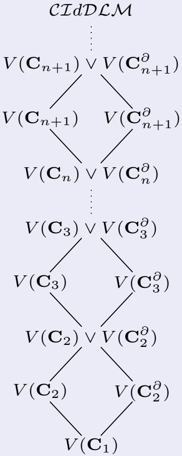


$$n \cdot m = \max_{\mathbb{N}}(n, m)$$

The subvariety lattice of $\mathcal{CI}d\mathcal{DLM}$

Theorem

The subvariety lattice of $\mathcal{CI}d\mathcal{DLM}$ *is of the form:*



Thank you!

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