Are finite affine topological spaces worthy of study?

Jeffrey T. Denniston[1] Jan Paseka[2] Sergejs Solovjovs[3]

[1]Kent State University, Kent, OH, USA, jdennist@kent.edu

[2]Masaryk University, Brno, Czech Republic, paseka@math.muni.cz

[3]Czech University of Life Sciences, Prague, Czech Republic, solovjovs@tf.czu.cz

Topology, Algebra, and Categories in Logic

University of Coimbra, Coimbra, Portugal
June 20 – 24, 2022

Acknowledgements

Jan Paseka acknowledges the support of the bilateral project "The many facets of orthomodularity" of the Austrian Science Fund (FWF) (project No. I 4579-N) and the Czech Science Foundation (GAČR) (project No. 20-09869L).

FWF Der Wissenschaftsfonds.

GAČR GRANTOVÁ AGENTURA ČESKÉ REPUBLIKY

Outline

1. Introduction
2. Affine spaces
3. Affine Sierpinski space
4. Affine Davey space
5. Conclusion

Davey space

- S. A. Morris (1984) showed that every topological space is homeomorphic to a subspace of a power of the *Davey space*, i.e., the space $\mathcal{D} = (D, \tau_D)$ having a 3-element underlying set $D = \{0, 1, 2\}$ and a topology $\tau_D = \{\varnothing, \{1\}, \{0, 1, 2\}\}$.
- Stating in a different language, $\mathcal{D}$ is an extremal coseparator in the category **Top** of topological spaces and continuous maps.
- In view of this result as well as to answer the criticism of some researchers claiming that "finite topological spaces are not in the slightest bit interesting", S. A. Morris stated that "perhaps there is something of interest in finite spaces after all".

Davey space

- S. A. Morris (1984) showed that every topological space is homeomorphic to a subspace of a power of the *Davey space*, i.e., the space $\mathcal{D} = (D, \tau_D)$ having a 3-element underlying set $D = \{0, 1, 2\}$ and a topology $\tau_D = \{\varnothing, \{1\}, \{0, 1, 2\}\}$.
- Stating in a different language, $\mathcal{D}$ is an extremal coseparator in the category **Top** of topological spaces and continuous maps.
- In view of this result as well as to answer the criticism of some researchers claiming that "finite topological spaces are not in the slightest bit interesting", S. A. Morris stated that "perhaps there is something of interest in finite spaces after all".

Davey space

- S. A. Morris (1984) showed that every topological space is homeomorphic to a subspace of a power of the *Davey space*, i.e., the space $\mathcal{D} = (D, \tau_D)$ having a 3-element underlying set $D = \{0, 1, 2\}$ and a topology $\tau_D = \{\varnothing, \{1\}, \{0, 1, 2\}\}$.
- Stating in a different language, $\mathcal{D}$ is an extremal coseparator in the category **Top** of topological spaces and continuous maps.
- In view of this result as well as to answer the criticism of some researchers claiming that "finite topological spaces are not in the slightest bit interesting", S. A. Morris stated that "perhaps there is something of interest in finite spaces after all".

Sierpinski space

- A topological space is T_0 if and only if it can be embedded into a power of the *Sierpinski space* $\mathcal{S} = (\{0,1\}, \{\varnothing, \{1\}, \{0,1\}\})$.
- Stating differently, $\mathcal{S}$ is an $\mathcal{M}$-coseparator in the category $\mathbf{Top}_0$ of T_0 topological spaces, where $\mathcal{M}$ stands for the class of topological embeddings (initial injective maps) in $\mathbf{Top}_0$.
- E. G. Manes (1974) introduced an analogue of the Sierpinski space for concrete categories called *Sierpinski object*.
- An object S of a concrete category $\mathbf{C}$ is a *Sierpinski object* if for every $\mathbf{C}$-object C, the hom-set $\mathbf{C}(C, S)$ is an initial source.

Sierpinski space

- A topological space is T_0 if and only if it can be embedded into a power of the *Sierpinski space* $\mathcal{S} = (\{0,1\}, \{\varnothing, \{1\}, \{0,1\}\})$.
- Stating differently, $\mathcal{S}$ is an $\mathcal{M}$-coseparator in the category $\mathbf{Top}_0$ of T_0 topological spaces, where $\mathcal{M}$ stands for the class of topological embeddings (initial injective maps) in $\mathbf{Top}_0$.
- E. G. Manes (1974) introduced an analogue of the Sierpinski space for concrete categories called *Sierpinski object*.
- An object S of a concrete category $\mathbf{C}$ is a *Sierpinski object* if for every $\mathbf{C}$-object C, the hom-set $\mathbf{C}(C, S)$ is an initial source.

Sierpinski space

- A topological space is T_0 if and only if it can be embedded into a power of the *Sierpinski space* $\mathcal{S} = (\{0,1\}, \{\varnothing, \{1\}, \{0,1\}\})$.
- Stating differently, $\mathcal{S}$ is an $\mathcal{M}$-coseparator in the category $\mathbf{Top}_0$ of T_0 topological spaces, where $\mathcal{M}$ stands for the class of topological embeddings (initial injective maps) in $\mathbf{Top}_0$.
- E. G. Manes (1974) introduced an analogue of the Sierpinski space for concrete categories called *Sierpinski object*.
- An object S of a concrete category $\mathbf{C}$ is a *Sierpinski object* if for every $\mathbf{C}$-object C, the hom-set $\mathbf{C}(C, S)$ is an initial source.

Sierpinski space

- A topological space is T_0 if and only if it can be embedded into a power of the *Sierpinski space* $\mathcal{S} = (\{0,1\}, \{\varnothing, \{1\}, \{0,1\}\})$.
- Stating differently, $\mathcal{S}$ is an $\mathcal{M}$-coseparator in the category $\mathbf{Top}_0$ of T_0 topological spaces, where $\mathcal{M}$ stands for the class of topological embeddings (initial injective maps) in $\mathbf{Top}_0$.
- E. G. Manes (1974) introduced an analogue of the Sierpinski space for concrete categories called *Sierpinski object*.
- An object S of a concrete category $\mathbf{C}$ is a *Sierpinski object* if for every $\mathbf{C}$-object C, the hom-set $\mathbf{C}(C, S)$ is an initial source.

Finite preorders

- G. Janelidze and M. Sobral (2002) showed that finite topological spaces are precisely the *finite preorders* (finite sets equipped with a reflexive and transitive binary relation).
- Nearly all the results of topological descent theory can be motivated by their finite instances, which become simple and natural when expressed in the language of finite preorders.

Finite preorders

- G. Janelidze and M. Sobral (2002) showed that finite topological spaces are precisely the *finite preorders* (finite sets equipped with a reflexive and transitive binary relation).
- Nearly all the results of topological descent theory can be motivated by their finite instances, which become simple and natural when expressed in the language of finite preorders.

Affine sets and affine topology

- There exists an affine approach to general topology, which is motivated by the notion of *affine set* of Y. Diers (1999).
- A classical topological space (X, τ) consists of a set X and a topology τ, where τ is a subset of the powerset $\mathcal{P}X$ of X and, moreover, τ has the algebraic structure of *frame*.
- The affine approach replaces the standard contravariant powerset functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{CBAlg}^{op}$ from the category $\mathbf{Set}$ of sets to the dual of the category of complete Boolean algebras with a functor $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$ from a category $\mathbf{X}$ to the dual category of a variety of algebras $\mathbf{A}$, requiring τ to be a subalgebra of TX.
- Taking suitable variety $\mathbf{A}$ and functor T, one gets not only the classical topological spaces, but also, e.g., the closure spaces and the most essential many-valued topological frameworks.

Affine sets and affine topology

- There exists an affine approach to general topology, which is motivated by the notion of *affine set* of Y. Diers (1999).
- A classical topological space (X, τ) consists of a set X and a topology τ, where τ is a subset of the powerset $\mathcal{P}X$ of X and, moreover, τ has the algebraic structure of *frame*.
- The affine approach replaces the standard contravariant powerset functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{CBAlg}^{op}$ from the category $\mathbf{Set}$ of sets to the dual of the category of complete Boolean algebras with a functor $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$ from a category $\mathbf{X}$ to the dual category of a variety of algebras $\mathbf{A}$, requiring τ to be a subalgebra of TX.
- Taking suitable variety $\mathbf{A}$ and functor T, one gets not only the classical topological spaces, but also, e.g., the closure spaces and the most essential many-valued topological frameworks.

Affine sets and affine topology

- There exists an affine approach to general topology, which is motivated by the notion of *affine set* of Y. Diers (1999).
- A classical topological space (X, τ) consists of a set X and a topology τ, where τ is a subset of the powerset $\mathcal{P}X$ of X and, moreover, τ has the algebraic structure of *frame*.
- The affine approach replaces the standard contravariant powerset functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{CBAlg}^{op}$ from the category **Set** of sets to the dual of the category of complete Boolean algebras with a functor $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$ from a category **X** to the dual category of a variety of algebras **A**, requiring τ to be a subalgebra of TX.
- Taking suitable variety **A** and functor T, one gets not only the classical topological spaces, but also, e.g., the closure spaces and the most essential many-valued topological frameworks.

Affine sets and affine topology

- There exists an affine approach to general topology, which is motivated by the notion of *affine set* of Y. Diers (1999).
- A classical topological space (X, τ) consists of a set X and a topology τ, where τ is a subset of the powerset $\mathcal{P}X$ of X and, moreover, τ has the algebraic structure of *frame*.
- The affine approach replaces the standard contravariant powerset functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{CBAlg}^{op}$ from the category **Set** of sets to the dual of the category of complete Boolean algebras with a functor $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$ from a category $\mathbf{X}$ to the dual category of a variety of algebras $\mathbf{A}$, requiring τ to be a subalgebra of TX.
- Taking suitable variety $\mathbf{A}$ and functor T, one gets not only the classical topological spaces, but also, e.g., the closure spaces and the most essential many-valued topological frameworks.

Aim of the talk

- This talk investigates the role of *finite* spaces in affine topology.
- There already exists an affine analogue of the Sierpinski space in terms of the Sierpinski object of E. G. Manes, which (in general) is no longer finite.
- This talk provides an affine analogue of the Davey space and shows its simple relation to the affine Sierpinski space.
- Since the affine Davey space is (in general) no longer finite as well, the talk conveys a message that finite spaces play a (probably) less important role in affine topological setting (e.g., in many-valued topology) than they do in the classical topology.

Aim of the talk

- This talk investigates the role of *finite* spaces in affine topology.
- There already exists an affine analogue of the Sierpinski space in terms of the Sierpinski object of E. G. Manes, which (in general) is no longer finite.
- This talk provides an affine analogue of the Davey space and shows its simple relation to the affine Sierpinski space.
- Since the affine Davey space is (in general) no longer finite as well, the talk conveys a message that finite spaces play a (probably) less important role in affine topological setting (e.g., in many-valued topology) than they do in the classical topology.

Aim of the talk

- This talk investigates the role of *finite* spaces in affine topology.
- There already exists an affine analogue of the Sierpinski space in terms of the Sierpinski object of E. G. Manes, which (in general) is no longer finite.
- This talk provides an affine analogue of the Davey space and shows its simple relation to the affine Sierpinski space.
- Since the affine Davey space is (in general) no longer finite as well, the talk conveys a message that finite spaces play a (probably) less important role in affine topological setting (e.g., in many-valued topology) than they do in the classical topology.

Aim of the talk

- This talk investigates the role of *finite* spaces in affine topology.
- There already exists an affine analogue of the Sierpinski space in terms of the Sierpinski object of E. G. Manes, which (in general) is no longer finite.
- This talk provides an affine analogue of the Davey space and shows its simple relation to the affine Sierpinski space.
- Since the affine Davey space is (in general) no longer finite as well, the talk conveys a message that finite spaces play a (probably) less important role in affine topological setting (e.g., in many-valued topology) than they do in the classical topology.

Ω-algebras and Ω-homomorphisms

Definition 1

Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a family of cardinal numbers, which is indexed by a (possibly proper or empty) class Λ.

- An *Ω-algebra* is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$, comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (*n_λ-ary primitive operations* on A).
- An *Ω-homomorphism* $(A_1, (\omega_\lambda^{A_1})_{\lambda \in \Lambda}) \xrightarrow{\varphi} (A_2, (\omega_\lambda^{A_2})_{\lambda \in \Lambda})$ is a map $A_1 \xrightarrow{\varphi} A_2$ such that $\varphi \circ \omega_\lambda^{A_1} = \omega_\lambda^{A_2} \circ \varphi^{n_\lambda}$ for every $\lambda \in \Lambda$.
- $\mathbf{Alg}(\Omega)$ is the construct of Ω-algebras and Ω-homomorphisms.

Forgetful functors of concrete categories will be denoted $| - |$.

Ω-algebras and Ω-homomorphisms

Definition 1

Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a family of cardinal numbers, which is indexed by a (possibly proper or empty) class Λ.

- An *Ω-algebra* is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$, comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (*n_λ-ary primitive operations* on A).
- An *Ω-homomorphism* $(A_1, (\omega_\lambda^{A_1})_{\lambda \in \Lambda}) \xrightarrow{\varphi} (A_2, (\omega_\lambda^{A_2})_{\lambda \in \Lambda})$ is a map $A_1 \xrightarrow{\varphi} A_2$ such that $\varphi \circ \omega_\lambda^{A_1} = \omega_\lambda^{A_2} \circ \varphi^{n_\lambda}$ for every $\lambda \in \Lambda$.
- $\mathbf{Alg}(\Omega)$ is the construct of Ω-algebras and Ω-homomorphisms.

Forgetful functors of concrete categories will be denoted $|-|$.

Varieties and algebras

Definition 2

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A *variety of Ω-algebras* is a full subcategory of $\mathbf{Alg}(\Omega)$, which is closed under the formation of products, $\mathcal{M}$-subobjects, and $\mathcal{E}$-quotients, and whose objects (resp. morphisms) are called *algebras* (resp. *homomorphisms*).

Example 3

1. **UQuant** is the variety of *unital quantales*.
2. **Frm** is the variety of *frames*.
3. **CBAlg** is the variety of *complete Boolean algebras*.
4. **CL** is the variety of *closure lattices* (*c-lattices*).

Varieties and algebras

Definition 2

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A *variety of Ω-algebras* is a full subcategory of $\mathbf{Alg}(\Omega)$, which is closed under the formation of products, $\mathcal{M}$-subobjects, and $\mathcal{E}$-quotients, and whose objects (resp. morphisms) are called *algebras* (resp. *homomorphisms*).

Example 3

1. **UQuant** is the variety of *unital quantales*.
2. **Frm** is the variety of *frames*.
3. **CBAlg** is the variety of *complete Boolean algebras*.
4. **CL** is the variety of *closure lattices* (*c-lattices*).

Affine spaces

Given a category $\mathbf{C}$, $\mathbf{C}^{op}$ stands for the dual category of $\mathbf{C}$.

Definition 4

Given a category $\mathbf{X}$, a variety of algebras $\mathbf{A}$, and a functor $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$, $\mathbf{Af\,Spc}(T)$ denotes the concrete category over $\mathbf{X}$, whose

objects (*T-affine spaces*) are pairs (X, τ), where X is an $\mathbf{X}$-object and τ is a subalgebra of TX;

morphisms (*T-affine morphisms*) $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ are $\mathbf{X}$-morphisms $X_1 \xrightarrow{f} X_2$ such that $(Tf)^{op}(\alpha) \in \tau_1$ for every $\alpha \in \tau_2$.

Affine spaces

Given a category $\mathbf{C}$, $\mathbf{C}^{op}$ stands for the dual category of $\mathbf{C}$.

Definition 4

Given a category $\mathbf{X}$, a variety of algebras $\mathbf{A}$, and a functor $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$, $\mathbf{Af\,Spc}(T)$ denotes the concrete category over $\mathbf{X}$, whose

objects (T*-affine spaces*) are pairs (X, τ), where X is an $\mathbf{X}$-object and τ is a subalgebra of TX;

morphisms (T*-affine morphisms*) $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ are $\mathbf{X}$-morphisms $X_1 \xrightarrow{f} X_2$ such that $(Tf)^{op}(\alpha) \in \tau_1$ for every $\alpha \in \tau_2$.

An example of functor T

Proposition 5

Given a subcategory $\mathbf{S}$ *of the category* $\mathbf{A}^{op}$, *there exists a functor* $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{P}_{\mathbf{S}}} \mathbf{A}^{op}$, $\mathcal{P}_{\mathbf{S}}((X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2)) = A_1^{X_1} \xrightarrow{\mathcal{P}_{\mathbf{S}}(f,\varphi)} A_2^{X_2}$, *where* $(\mathcal{P}_{\mathbf{S}}(f,\varphi))^{op}(\alpha) = \varphi^{op} \circ \alpha \circ f$ *for every* $\alpha \in A_2^{X_2}$.

$\mathbf{S} = \{A \xrightarrow{1_A} A\}$ provides a functor $\mathbf{Set} \xrightarrow{\mathcal{P}_A} \mathbf{A}^{op}$, $\mathcal{P}_A(X_1 \xrightarrow{f} X_2) = A^{X_1} \xrightarrow{\mathcal{P}_A f} A^{X_2}$, where $(\mathcal{P}_A f)^{op}(\alpha) = \alpha \circ f$ for every $\alpha \in A_2^{X_2}$.

Example 6

$\mathbf{A} = \mathbf{CBAlg}$ and $\mathbf{S} = \{2 \xrightarrow{1_2} 2\}$ provide the classical contravariant powerset functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{CBAlg}^{op}$, defined on a map $X_1 \xrightarrow{f} X_2$ by $\mathcal{P}X_2 \xrightarrow{(\mathcal{P}f)^{op}} \mathcal{P}X_1$, where $(\mathcal{P}f)^{op}(S) = \{x \in X_1 \mid f(x) \in S\}$.

An example of functor T

Proposition 5

Given a subcategory $\mathbf{S}$ *of the category* $\mathbf{A}^{op}$*, there exists a functor* $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{P}_{\mathbf{S}}} \mathbf{A}^{op}$, $\mathcal{P}_{\mathbf{S}}((X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2)) = A_1^{X_1} \xrightarrow{\mathcal{P}_{\mathbf{S}}(f,\varphi)} A_2^{X_2}$, *where* $(\mathcal{P}_{\mathbf{S}}(f,\varphi))^{op}(\alpha) = \varphi^{op} \circ \alpha \circ f$ *for every* $\alpha \in A_2^{X_2}$.

$\mathbf{S} = \{A \xrightarrow{1_A} A\}$ provides a functor $\mathbf{Set} \xrightarrow{\mathcal{P}_A} \mathbf{A}^{op}$, $\mathcal{P}_A(X_1 \xrightarrow{f} X_2) = A^{X_1} \xrightarrow{\mathcal{P}_A f} A^{X_2}$, where $(\mathcal{P}_A f)^{op}(\alpha) = \alpha \circ f$ for every $\alpha \in A_2^{X_2}$.

Example 6

$\mathbf{A} = \mathbf{CBAlg}$ and $\mathbf{S} = \{2 \xrightarrow{1_2} 2\}$ provide the classical contravariant powerset functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{CBAlg}^{op}$, defined on a map $X_1 \xrightarrow{f} X_2$ by $\mathcal{P}X_2 \xrightarrow{(\mathcal{P}f)^{op}} \mathcal{P}X_1$, where $(\mathcal{P}f)^{op}(S) = \{x \in X_1 \mid f(x) \in S\}$.

An example of functor T

Proposition 5

Given a subcategory $\mathbf{S}$ *of the category* $\mathbf{A}^{op}$*, there exists a functor* $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{P}_{\mathbf{S}}} \mathbf{A}^{op}$, $\mathcal{P}_{\mathbf{S}}((X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2)) = A_1^{X_1} \xrightarrow{\mathcal{P}_{\mathbf{S}}(f,\varphi)} A_2^{X_2}$, *where* $(\mathcal{P}_{\mathbf{S}}(f,\varphi))^{op}(\alpha) = \varphi^{op} \circ \alpha \circ f$ *for every* $\alpha \in A_2^{X_2}$.

$\mathbf{S} = \{A \xrightarrow{1_A} A\}$ provides a functor $\mathbf{Set} \xrightarrow{\mathcal{P}_A} \mathbf{A}^{op}$, $\mathcal{P}_A(X_1 \xrightarrow{f} X_2) = A^{X_1} \xrightarrow{\mathcal{P}_A f} A^{X_2}$, where $(\mathcal{P}_A f)^{op}(\alpha) = \alpha \circ f$ for every $\alpha \in A_2^{X_2}$.

Example 6

$\mathbf{A} = \mathbf{CBAlg}$ and $\mathbf{S} = \{2 \xrightarrow{1_2} 2\}$ provide the classical contravariant powerset functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{CBAlg}^{op}$, defined on a map $X_1 \xrightarrow{f} X_2$ by $\mathcal{P}X_2 \xrightarrow{(\mathcal{P}f)^{op}} \mathcal{P}X_1$, where $(\mathcal{P}f)^{op}(S) = \{x \in X_1 \mid f(x) \in S\}$.

Examples of affine spaces

Example 7

1. If $\mathbf{A} = \mathbf{Frm}$, then $\mathbf{Af\,Spc}(\mathcal{P}_2)$ is the category **Top** of topological spaces.
2. If $\mathbf{A}$ is a variety of algebras, then $\mathbf{Af\,Spc}(\mathcal{P}_A)$ is the category $\mathbf{ASet}(\Omega)$ of affine sets of E. Giuli and D. Hofmann (2009).
3. If $\mathbf{A} = \mathbf{UQuant}$ or $\mathbf{A} = \mathbf{Frm}$, then $\mathbf{Af\,Spc}(\mathcal{P}_{\mathbf{S}})$ is the category **S-Top** of variable-basis many-valued topological spaces of S. E. Rodabaugh (1999, 2007).
4. If $\mathbf{A} = \mathbf{CL}$, then $\mathbf{Af\,Spc}(\mathcal{P}_2)$ is the category **Cls** of closure spaces of D. Aerts *et al.* (1999).

Properties of the category of affine spaces

Given an algebra A of a variety $\mathbf{A}$ and a subset $S \subseteq A$, $\langle S \rangle$ stands for the subalgebra of A generated by the set S.

Theorem 8

The concrete category $(\mathbf{Af\,Spc}(T), |-|)$ is topological over $\mathbf{X}$.

Proof.

Given a $|-|$-structured source $\mathcal{L} = (X \xrightarrow{f_i} |(X_i, \tau_i)|)_{i \in I}$, the initial structure on X w.r.t. $\mathcal{L}$ can be defined by $\tau = \langle \bigcup_{i \in I} (Tf_i)^{op}(\tau_i) \rangle$.

Corollary 9

If $\mathbf{X}$ has (co)products, then $\mathbf{Af\,Spc}(T)$ has concrete (co)products.

Properties of the category of affine spaces

Given an algebra A of a variety $\mathbf{A}$ and a subset $S \subseteq A$, $\langle S \rangle$ stands for the subalgebra of A generated by the set S.

Theorem 8

The concrete category $(\mathbf{Af\,Spc}(T), |-|)$ *is topological over* $\mathbf{X}$.

Proof.

Given a $|-|$-structured source $\mathcal{L} = (X \xrightarrow{f_i} |(X_i, \tau_i)|)_{i \in I}$, the initial structure on X w.r.t. $\mathcal{L}$ can be defined by $\tau = \langle \bigcup_{i \in I} (Tf_i)^{op}(\tau_i) \rangle$.

Corollary 9

If $\mathbf{X}$ *has (co)products, then* $\mathbf{Af\,Spc}(T)$ *has concrete (co)products.*

Properties of the category of affine spaces

Given an algebra A of a variety $\mathbf{A}$ and a subset $S \subseteq A$, $\langle S \rangle$ stands for the subalgebra of A generated by the set S.

Theorem 8

The concrete category $(\mathbf{Af\,Spc}(T), |-|)$ *is topological over* $\mathbf{X}$.

Proof.

Given a $|-|$-structured source $\mathcal{L} = (X \xrightarrow{f_i} |(X_i, \tau_i)|)_{i \in I}$, the initial structure on X w.r.t. $\mathcal{L}$ can be defined by $\tau = \langle \bigcup_{i \in I} (Tf_i)^{op}(\tau_i) \rangle$.

Corollary 9

If $\mathbf{X}$ *has (co)products, then* $\mathbf{Af\,Spc}(T)$ *has concrete (co)products.*

Properties of the category of affine spaces

Given an algebra A of a variety $\mathbf{A}$ and a subset $S \subseteq A$, $\langle S \rangle$ stands for the subalgebra of A generated by the set S.

Theorem 8

The concrete category $(\mathbf{Af\,Spc}(T), |-|)$ *is topological over* $\mathbf{X}$.

Proof.

Given a $|-|$-structured source $\mathcal{L} = (X \xrightarrow{f_i} |(X_i, \tau_i)|)_{i \in I}$, the initial structure on X w.r.t. $\mathcal{L}$ can be defined by $\tau = \langle \bigcup_{i \in I} (Tf_i)^{op}(\tau_i) \rangle$.

Corollary 9

If $\mathbf{X}$ *has (co)products, then* $\mathbf{Af\,Spc}(T)$ *has concrete (co)products.*

Employed setting

- Fix a variety of algebras $\mathbf{A}$, an $\mathbf{A}$-algebra L, and consider the category $\mathbf{Af\,Spc}(\mathcal{P}_L)$ denoted $\mathbf{Af\,Spc}(L)$, whose objects and morphisms will be called *affine spaces* and *affine morphisms*.
- Assume that the fixed algebra L has at least two elements, which excludes the trivial cases of the empty algebra (provided that it exists in the variety $\mathbf{A}$) and a singleton algebra.

Employed setting

- Fix a variety of algebras $\mathbf{A}$, an $\mathbf{A}$-algebra L, and consider the category $\mathbf{Af\,Spc}(\mathcal{P}_L)$ denoted $\mathbf{Af\,Spc}(L)$, whose objects and morphisms will be called *affine spaces* and *affine morphisms*.
- Assume that the fixed algebra L has at least two elements, which excludes the trivial cases of the empty algebra (provided that it exists in the variety $\mathbf{A}$) and a singleton algebra.

Affine Sierpinski space

Definition 10

Affine Sierpinski space is the pair $\mathsf{S} = (|L|, \langle 1_L \rangle)$, where $\langle 1_L \rangle$ is the subalgebra of $L^{|L|}$ generated by the identity map 1_L.

Example 11

$\mathbf{A} = \mathbf{Frm}$ and $L = 2$ provide the classical *Sierpinski space* $\mathcal{S} = (\{0,1\}, \{\varnothing, \{1\}, \{0,1\}\})$.

Affine Sierpinski space

Definition 10

Affine Sierpinski space is the pair $\mathsf{S} = (|L|, \langle 1_L \rangle)$, where $\langle 1_L \rangle$ is the subalgebra of $L^{|L|}$ generated by the identity map 1_L.

Example 11

$\mathbf{A} = \mathbf{Frm}$ and $L = 2$ provide the classical *Sierpinski space* $\mathcal{S} = (\{0, 1\}, \{\varnothing, \{1\}, \{0, 1\}\})$.

Properties of affine Sierpinski space

Definition 12

An affine space (X, τ) is said to be T_0 provided that for every distinct $x_1, x_2 \in X$, there exists $\alpha \in \tau$ such that $\alpha(x_1) \neq \alpha(x_2)$.

Theorem 13

An affine space (X, τ) is T_0 iff it is embeddable into a power of S.

Corollary 14

S is an $\mathcal{M}$-coseparator in the category $\mathbf{Af\,Spc}_0(L)$ of T_0 affine spaces, where $\mathcal{M}$ is the class of embeddings in $\mathbf{Af\,Spc}_0(L)$.

Properties of affine Sierpinski space

Definition 12

An affine space (X, τ) is said to be T_0 provided that for every distinct $x_1, x_2 \in X$, there exists $\alpha \in \tau$ such that $\alpha(x_1) \neq \alpha(x_2)$.

Theorem 13

An affine space (X, τ) is T_0 iff it is embeddable into a power of S.

Corollary 14

S *is an $\mathcal{M}$-coseparator in the category $\mathbf{Af\,Spc}_0(L)$ of T_0 affine spaces, where $\mathcal{M}$ is the class of embeddings in $\mathbf{Af\,Spc}_0(L)$.*

Properties of affine Sierpinski space

Definition 12

An affine space (X, τ) is said to be T_0 provided that for every distinct $x_1, x_2 \in X$, there exists $\alpha \in \tau$ such that $\alpha(x_1) \neq \alpha(x_2)$.

Theorem 13

An affine space (X, τ) is T_0 iff it is embeddable into a power of S.

Corollary 14

S *is an $\mathcal{M}$-coseparator in the category $\mathbf{Af\,Spc}_0(L)$ of T_0 affine spaces, where $\mathcal{M}$ is the class of embeddings in $\mathbf{Af\,Spc}_0(L)$.*

Finite affine Sierpinski space

- The cardinality of the underlying set of the affine Sierpinski space depends on the cardinality of the algebra L, namely, it can be arbitrarily large.
- For the variety **Frm** of frames, replacing the two-element frame 2 with an infinite frame (to get *an infinite set of truth values* in many-valued topology) gives an infinite affine Sierpinski space.

Message

It is not the underlying set of a topological space, which plays the main role, but rather the algebra underlying the respective powerset.

Finite affine Sierpinski space

- The cardinality of the underlying set of the affine Sierpinski space depends on the cardinality of the algebra L, namely, it can be arbitrarily large.
- For the variety **Frm** of frames, replacing the two-element frame 2 with an infinite frame (to get *an infinite set of truth values* in many-valued topology) gives an infinite affine Sierpinski space.

Message

It is not the underlying set of a topological space, which plays the main role, but rather the algebra underlying the respective powerset.

Finite affine Sierpinski space

- The cardinality of the underlying set of the affine Sierpinski space depends on the cardinality of the algebra L, namely, it can be arbitrarily large.
- For the variety **Frm** of frames, replacing the two-element frame 2 with an infinite frame (to get *an infinite set of truth values* in many-valued topology) gives an infinite affine Sierpinski space.

Message

It is not the underlying set of a topological space, which plays the main role, but rather the algebra underlying the respective powerset.

An assumption on the underlying variety

Assumption 15

The variety $\mathbf{A}$ *has at least one nullary operation* ω_0 *(namely, a constant, which is an element of every algebra of the variety* $\mathbf{A}$*).*

ω_0^L will denote the respective constant in the fixed algebra L.

Example 16

The varieties **UQuant**, **Frm**, **CBAlg**, and **CL** satisfy Assumption 15.

An assumption on the underlying variety

Assumption 15

The variety $\mathbf{A}$ *has at least one nullary operation* ω_0 *(namely, a constant, which is an element of every algebra of the variety* $\mathbf{A}$*).*

ω_0^L will denote the respective constant in the fixed algebra L.

Example 16

The varieties **UQuant**, **Frm**, **CBAlg**, and **CL** satisfy Assumption 15.

An assumption on the underlying variety

Assumption 15

The variety $\mathbf{A}$ *has at least one nullary operation* ω_0 *(namely, a constant, which is an element of every algebra of the variety* $\mathbf{A}$*).*

ω_0^L will denote the respective constant in the fixed algebra L.

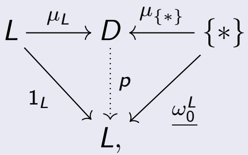
Example 16

The varieties **UQuant**, **Frm**, **CBAlg**, and **CL** satisfy Assumption 15.

Affine Davey space

Definition 17

Affine Davey space is the pair $\mathsf{D} = (D, \tau_D)$, in which $D = |L| \coprod \{*\}$, where "$\coprod$" stands for the coproduct in the category **Set**, and it is, moreover, assumed that $* \notin L$, and $\tau_D = \langle p \rangle \subseteq L^D$, in which the map $D \xrightarrow{p} L$ is given by the following commutative diagram:



where μ_L and $\mu_{\{*\}}$ are the coproduct injections, and $\underline{\omega_0^L}(*) = \omega_0^L$.

Examples of affine Davey space

Example 18

Since every frame has two nullary operations, i.e., the bottom element 0 and the top element 1, the classical case of $\mathbf{A} = \mathbf{Frm}$ and $L = 2 = \{0, 1\}$ provides the following topological spaces D:

1. Taking $\omega_0^2 = 0$, one obtains the 3-element set $D = \{0, 1, *\}$ and the topology $\tau_D = \{\varnothing, \{1\}, \{0, 1, *\}\}$, namely, D is precisely the classical Davey space $\mathcal{D}$ of S. A. Morris.
2. Taking $\omega_0^2 = 1$, one obtains the 3-element set $D = \{0, 1, *\}$ and the topology $\tau_D = \{\varnothing, \{1, *\}, \{0, 1, *\}\}$, namely, D is the second possible form of the Davey space $\mathcal{D}$ of S. A. Morris.

Properties of affine Davey space

Theorem 19

Every affine space can be embedded into a power of D.

Proof.

- For an affine space (X,τ), let $K=\{\alpha \mid \alpha\in\tau\}$ and $J=K\bigcup X$.
- For every $\alpha\in K$, define a map $X \xrightarrow{f_\alpha} D := X \xrightarrow{\alpha} L \xrightarrow{\mu_L} D$, and show that $|(X,\tau)| \xrightarrow{f_\alpha} |\mathrm{D}|$ is an affine morphism.
- For every $x\in X$, define a map $X \xrightarrow{f_x} D$ by

$$f_x(y)=\begin{cases} *, & y=x\\ \omega_0^L, & \text{otherwise},\end{cases}$$

and show that $|(X,\tau)| \xrightarrow{f_x} |\mathrm{D}|$ is an affine morphism.

Properties of affine Davey space

Theorem 19

Every affine space can be embedded into a power of D.

Proof.

- For an affine space (X,τ), let $K=\{\alpha \mid \alpha\in\tau\}$ and $J=K\bigcup X$.
- For every $\alpha\in K$, define a map $X\xrightarrow{f_\alpha} D := X\xrightarrow{\alpha} L\xrightarrow{\mu_L} D$, and show that $|(X,\tau)|\xrightarrow{f_\alpha}|\mathrm{D}|$ is an affine morphism.
- For every $x\in X$, define a map $X\xrightarrow{f_x} D$ by

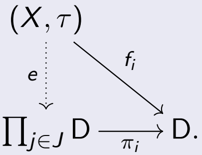
$$f_x(y)=\begin{cases} *, & y=x\\ \omega_0^L, & \text{otherwise},\end{cases}$$

and show that $|(X,\tau)|\xrightarrow{f_x}|\mathrm{D}|$ is an affine morphism.

Properties of affine Davey space

Proof cont.

- The above maps provide an affine morphism $(X,\tau) \xrightarrow{e} \prod_{j\in J} \mathrm{D}$, defined by the following commutative (for every $i \in J$) diagram:



- Show that the affine morphism e is an embedding.

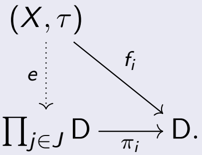
Corollary 20

D is an extremal coseparator in the category $\mathbf{Af\,Spc}(L)$.

Properties of affine Davey space

Proof cont.

- The above maps provide an affine morphism $(X, \tau) \xrightarrow{e} \prod_{j \in J} \mathrm{D}$, defined by the following commutative (for every $i \in J$) diagram:



- Show that the affine morphism e is an embedding.

Corollary 20

D *is an extremal coseparator in the category* **Af Spc**(L).

Finite affine Davey space

- The cardinality of the underlying set of the affine Davey space depends on the cardinality of the algebra L, namely, it can be arbitrarily large.
- For the variety **Frm** of frames, replacing the two-element frame 2 underlying the classical topology with an infinite frame (e.g., taking the unit interval $[0, 1]$ as the set of *truth values* for many-valued topology) provides an infinite affine Davey space.

Message

It is not the underlying set of a topological space, which plays the main role, but rather the algebra underlying the respective powerset.

Finite affine Davey space

- The cardinality of the underlying set of the affine Davey space depends on the cardinality of the algebra L, namely, it can be arbitrarily large.
- For the variety **Frm** of frames, replacing the two-element frame 2 underlying the classical topology with an infinite frame (e.g., taking the unit interval $[0, 1]$ as the set of *truth values* for many-valued topology) provides an infinite affine Davey space.

Message

It is not the underlying set of a topological space, which plays the main role, but rather the algebra underlying the respective powerset.

Finite affine Davey space

- The cardinality of the underlying set of the affine Davey space depends on the cardinality of the algebra L, namely, it can be arbitrarily large.
- For the variety **Frm** of frames, replacing the two-element frame 2 underlying the classical topology with an infinite frame (e.g., taking the unit interval $[0, 1]$ as the set of *truth values* for many-valued topology) provides an infinite affine Davey space.

Message

It is not the underlying set of a topological space, which plays the main role, but rather the algebra underlying the respective powerset.

Affine Davey space versus affine Sierpinski space

Definition 21

An affine space (X, τ) is said to be *indiscrete* provided that $\tau = \langle \varnothing \rangle$.

Proposition 22

D contains S and an indiscrete 2-element space as a subspace.

Proof.

Observe that $Z = \{\omega_0^L, *\}$ is a 2-element indiscrete subspace of D.

Proposition 23

Every indiscrete subspace of D has at most two elements.

Affine Davey space versus affine Sierpinski space

Definition 21

An affine space (X, τ) is said to be *indiscrete* provided that $\tau = \langle\varnothing\rangle$.

Proposition 22

D *contains* S *and an indiscrete 2-element space as a subspace.*

Proof.

Observe that $Z = \{\omega_0^L, *\}$ is a 2-element indiscrete subspace of D.

Proposition 23

Every indiscrete subspace of D *has at most two elements.*

Affine Davey space versus affine Sierpinski space

Definition 21

An affine space (X, τ) is said to be *indiscrete* provided that $\tau = \langle \varnothing \rangle$.

Proposition 22

D *contains* S *and an indiscrete 2-element space as a subspace.*

Proof.

Observe that $Z = \{\omega_0^L, *\}$ is a 2-element indiscrete subspace of D.

Proposition 23

Every indiscrete subspace of D *has at most two elements.*

Affine Davey space versus affine Sierpinski space

Definition 21

An affine space (X, τ) is said to be *indiscrete* provided that $\tau = \langle \varnothing \rangle$.

Proposition 22

D *contains* S *and an indiscrete 2-element space as a subspace.*

Proof.

Observe that $Z = \{\omega_0^L, *\}$ is a 2-element indiscrete subspace of D.

Proposition 23

Every indiscrete subspace of D *has at most two elements.*

Final remarks: affine Davey space

- Motivated by the result of S. A. Morris, stating that every topological space is homeomorphic to a subspace of a product of copies of the Davey space $\mathcal{D} = (\{0, 1, 2\}, \{\varnothing, \{1\}, \{0, 1, 2\}\})$, this talk provided an analogue of this result for affine topological spaces based in the notion of affine set of Y. Diers.
- Similar to the classical topology, the affine Davey space is an extremal coseparator in the category of affine topological spaces, as well as contains the affine Sierpinski space and an indiscrete 2-element space as a subspace.
- While the classical Davey space is finite, which generates interest in finite topological spaces, its affine analogue can have an arbitrarily large cardinality, which depends on the cardinality of the algebra underlying the respective powersets.

Final remarks: affine Davey space

- Motivated by the result of S. A. Morris, stating that every topological space is homeomorphic to a subspace of a product of copies of the Davey space $\mathcal{D} = (\{0, 1, 2\}, \{\varnothing, \{1\}, \{0, 1, 2\}\})$, this talk provided an analogue of this result for affine topological spaces based in the notion of affine set of Y. Diers.
- Similar to the classical topology, the affine Davey space is an extremal coseparator in the category of affine topological spaces, as well as contains the affine Sierpinski space and an indiscrete 2-element space as a subspace.
- While the classical Davey space is finite, which generates interest in finite topological spaces, its affine analogue can have an arbitrarily large cardinality, which depends on the cardinality of the algebra underlying the respective powersets.

Final remarks: affine Davey space

- Motivated by the result of S. A. Morris, stating that every topological space is homeomorphic to a subspace of a product of copies of the Davey space $\mathcal{D} = (\{0,1,2\},\{\varnothing,\{1\},\{0,1,2\}\})$, this talk provided an analogue of this result for affine topological spaces based in the notion of affine set of Y. Diers.
- Similar to the classical topology, the affine Davey space is an extremal coseparator in the category of affine topological spaces, as well as contains the affine Sierpinski space and an indiscrete 2-element space as a subspace.
- While the classical Davey space is finite, which generates interest in finite topological spaces, its affine analogue can have an arbitrarily large cardinality, which depends on the cardinality of the algebra underlying the respective powersets.

Final remarks: finite affine spaces

- Since both affine Davey and Sierpinski space, being an extremal and an $\mathcal{M}$-coseparator ($\mathcal{M}$ is the class of embeddings) in the categories of affine spaces and T_0 affine spaces, respectively, can have arbitrarily large cardinalities, this talk claims that in affine topology (e.g., in many-valued topology) finite spaces no longer play such a big role as they do in the classical topology.
- The switch from the classical "true" and "false" truth values (as in the *classical logic*) to an infinite number of truth values (as in some *many-valued logics*) brings with it the necessity to include all of them into the underlying sets of the respective Davey and Sierpinski spaces that makes these spaces infinite.

Final remarks: finite affine spaces

- Since both affine Davey and Sierpinski space, being an extremal and an $\mathcal{M}$-coseparator ($\mathcal{M}$ is the class of embeddings) in the categories of affine spaces and T_0 affine spaces, respectively, can have arbitrarily large cardinalities, this talk claims that in affine topology (e.g., in many-valued topology) finite spaces no longer play such a big role as they do in the classical topology.
- The switch from the classical "true" and "false" truth values (as in the *classical logic*) to an infinite number of truth values (as in some *many-valued logics*) brings with it the necessity to include all of them into the underlying sets of the respective Davey and Sierpinski spaces that makes these spaces infinite.

An open problem

Problem 24

Provide an explicit description of affine Davey and Sierpinski spaces in the general category $\mathbf{Af\,Spc}(T)$, *namely, replacing the functor* $\mathbf{Set} \xrightarrow{\mathcal{P}_L} \mathbf{A}^{op}$ *of this talk with a general functor* $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$.

References I

J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and Concrete Categories: the Joy of Cats*, Repr. Theory Appl. Categ. **17** (2006), 1–507.

D. Aerts, E. Colebunders, A. van der Voorde, and B. van Steirteghem, *State property systems and closure spaces: a study of categorical equivalence*, Int. J. Theor. Phys. **38** (1999), no. 1, 359–385.

Y. Diers, *Affine algebraic sets relative to an algebraic theory*, J. Geom. **65** (1999), no. 1-2, 54–76.

E. Giuli and D. Hofmann, *Affine sets: the structure of complete objects and duality*, Topology Appl. **156** (2009), no. 12, 2129–2136.

G. Janelidze and M. Sobral, *Finite preorders and topological descent I*, J. Pure Appl. Algebra **175** (2002), no. 1-3, 187–205.

References II

E. G. Manes, *Compact Hausdorff objects*, General Topology Appl. **4** (1974), 341–360.

S. A. Morris, *Are finite topological spaces worthy of study?*, Aust. Math. Soc. Gaz. **11** (1984), 31–32.

S. E. Rodabaugh, *Categorical Foundations of Variable-Basis Fuzzy Topology*, Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory (U. Höhle and S. E. Rodabaugh, eds.), Dordrecht: Kluwer Academic Publishers, 1999, pp. 273–388.

S. E. Rodabaugh, *Relationship of Algebraic Theories to Powerset Theories and Fuzzy Topological Theories for Lattice-Valued Mathematics*, Int. J. Math. Math. Sci. **2007** (2007), 1–71.

S. Solovyov, *Categorical foundations of variety-based topology and topological systems*, Fuzzy Sets Syst. **192** (2012), 176–200.

Thank you for your attention!