Coextensive varieties and the Gaeta topos.

W. J. Zuluaga Botero

Departamento de Matemática,
Universidad Nacional del Centro de la Provincia de Buenos Aires,
Tandil–Argentina

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Theorem

Let $\mathcal{V}$ be a variety with BFC. The map $g : Z(A) \to FC(A)$, defined by $g(\vec{e}) = \theta^A_{\vec{0},\vec{e}}$ is a bijection and its inverse $h : FC(A) \to Z(A)$ is defined by $h(\theta) = \vec{e}$, where $\vec{e}$ is the only $\vec{e} \in A^N$ such that $\vec{e}/\theta = \vec{0}/\theta$ and $\vec{e}/\theta^* = \vec{1}/\theta^*$.

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$\mathbf{Z}(\mathrm{A}) = (Z(\mathrm{A}), \vee_{\mathrm{A}}, \wedge_{\mathrm{A}}, ^{c_{\mathrm{A}}}, \vec{0}, \vec{1})$ is a Boolean algebra which is isomorphic to $(FC(\mathrm{A}), \vee, \cap, ^{*}, \Delta^{\mathrm{A}}, \nabla^{\mathrm{A}})$.

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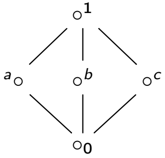

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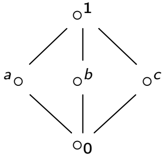

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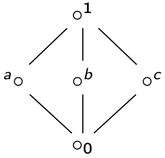

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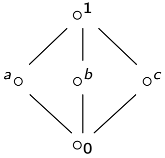

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- Church varieties: Varieties with a term $u(x, y, z)$ and 0-ary terms 0 and 1 satisfying $u(x, y, 0) = x$ and $u(x, y, 1) = y$.

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Let E be a topos. A $\mathcal{V}$-model X of E is *$\mathcal{V}$-indecomposable* if the sequents

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Proposition

Let E be a topos. A $\mathcal{V}$-model X of E is *$\mathcal{V}$-indecomposable* iff the diagram below

$$0 \xrightarrow{!} 1 \underset{\vec{0}}{\overset{\vec{1}}{\rightrightarrows}} X^n$$

is an equalizer in E, and the morphism $\alpha : 1 + 1 \rightarrow [\sigma(\vec{x}, \vec{y})]_X$ is an isomorphism.

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Proposition

Let $\mathcal{V}$ be a coextensive variety of finite type. If $\mathcal{V}$ is locally finite then it is fp-coextensive.

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(2) $\mathsf{F}_{\mathcal{V}}(x)$ *is indecomposable.*

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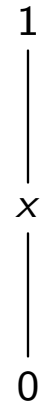

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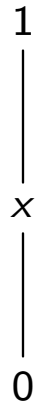

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Integral rigs

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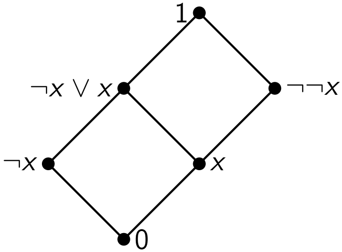

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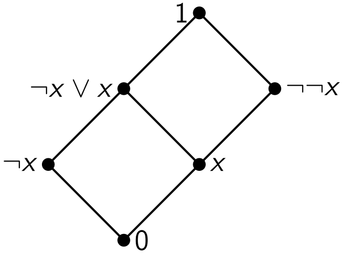
- $\mathcal{PH}$ is fp-coextensive. (it is locally finite)
- $\mathsf{F}_{\mathcal{PH}}(x)$ is decomposable. ($\neg x \vee \neg\neg x = 1$ and $\neg x \wedge \neg\neg x = 0$)



$\therefore$ $\mathcal{G}(\mathcal{PH})$ does not classifies $\mathcal{MV}$-indecomposable objects.

Thanks !

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